

NONLINEAR OBSERVERS

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Summary

This paper is a review of the existing methods for designing an observer for a system modeled by nonlinear equations. We focus our attention on autonomous, finite dimensional systems described by ordinary differential equations. The current condition of such a system is described by its state variables about which we just have partial and possibly noisy measurements. The goal of the observer is to process these measurements and any information regarding the initial state of the system to obtain an estimate of the current state of the system. This estimate should improve with additional measurements and, ideally, converge to the true value in the absence of noise. The observer does this by taking advantage of our *a priori* knowledge of the dynamics of the system.

1. Introduction

Systems are sets of components, physical or otherwise, which are connected in such a manner as to form and act as entire units. A nonlinear system is described by a mathematical model consisting of inputs, states, and outputs whose dynamics is given by nonlinear equations. Such models are used to represent a wide variety of dynamic processes in the real world. The inputs are the way the external world affects the system, the states are the internal memory of the system and the outputs are the way the system affects the external world. An example of such a system is

$$\dot{x}(t) = f(t, x(t), u(t)) \quad (1)$$

$$y(t) = h(t, x(t), u(t)) \quad (2)$$

$$x(0) \approx \hat{x}^0. \quad (3)$$

The input is the m vector u , the state is the n vector x and the output is the p vector y . The state of the system at the initial time $t = 0$ is not known exactly but is approximately \hat{x}^0 . Typically, the dimensions of the input and output are less than that of the state.

A particular case is an autonomous linear system

$$\dot{x} = Ax + Bu \quad (4)$$

$$y = Cx + Du \quad (5)$$

$$x(0) \approx \hat{x}^0. \quad (6)$$

Other examples include systems described by difference equations

$$x(t+1) = f(t, x(t), u(t)) \quad (7)$$

$$y(t) = h(t, x(t), u(t)) \quad (8)$$

and infinite dimensional systems described by partial differential and/or difference equations, delay differential equations or integro-differential equations. This review will focus on finite dimensional systems described by ordinary differential equations.

An observer is a method of estimating the state of the system from partial and possibly noisy measurements of the inputs and outputs and inexact knowledge of the initial condition. More precisely an observer is a causal mapping from any prior information about the initial state x^0 and from the past inputs and outputs

$$\{(u(\tau), y(\tau)) : t^0 \leq \tau \leq t\} \quad (9)$$

to an estimate $\hat{x}(t)$ of the current state $x(t)$ or an estimate $\hat{z}(t)$ of some function $z(t) = \kappa(x(t))$ of the current state. Causality means that the estimate at time t does not depend on any information about the inputs and outputs after time t . This restriction reflects the need to use the estimate in real time to control the system. The essential requirement of an observer is that the estimate converges to the true value as t gets large.

Sometimes it is not necessary to estimate the full state but only some function of it, say $\kappa(t, x)$. For example, if one wishes to use the feedback control $u = \kappa(t, x)$. This article will focus on observers of the full state.

The prototype of an observer is that of an autonomous linear system Eqs. (4) - (6). The

system

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \quad (10)$$

$$\hat{y} = C\hat{x} + Du \quad (11)$$

$$\hat{x}(0) = \hat{x}^0 \quad (12)$$

is an observer where L is an $n \times p$ matrix to be chosen by the designer. The dynamics of the error $\tilde{x} = x - \hat{x}$ is given by

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \quad (13)$$

$$\tilde{x}(0) = x^0 - \hat{x}^0 \quad (14)$$

If the spectrum of the matrix $A - LC$ lies in the open left half plane, then the error decays to zero exponentially fast. In this way, the problem of designing an observer for an autonomous linear system is reduced to the following problem. Given A , C , find L so that $A - LC$ is Hurwitz, i. e., the spectrum of $A - LC$ is in the open left half plane. We discuss when L can be so chosen in the next section (see *Design Techniques for Time Varying Systems* for further details.)

For nonlinear systems the distinction between nonautonomous Eqs. (1) - (3) and autonomous systems

$$\dot{x} = f(x, u) \quad (15)$$

$$y = h(x, u) \quad (16)$$

$$x(0) = \hat{x}^0 \quad (17)$$

is frequently not important as one can add time as an extra state $x_{n+1} = t - t^0$ and thereby reduce the former to the latter. Since an observer operates in real time, time is usually observable and so can be added as an extra output also. Frequently models depend on parameters θ , as in $\dot{x} = f(x, u, \theta)$. But in a nonlinear system the distinction between states and parameters is not always clearcut. Parameters can always be treated as additional states by adding the differential equation $\dot{\theta} = 0$. Therefore, the problem of real time parameter estimation reduces to the problem of real time state estimation and may be solvable by an observer. If the state estimate is not going to be used in real time, then one can collect data after time t to estimate $x(t)$. This problem is sometimes called nonlinear smoothing and is related to the identification of nonlinear systems (see *Identification of Nonlinear Systems*).

Another example of an observer is the extended Kalman filter described in more detail

in (see *State Reconstruction by Extended Kalman Filter*) and in the following statements. This is an observer for a nonlinear, nonautonomous system Eqs. (1) - (3) which is derived using stochastic arguments. Two quantities $\hat{x}(t)$ and $P(t)$ are computed by the extended Kalman filter. The stochastic interpretation is that the distribution of the true state $x(t)$ is approximately Gaussian with mean $\hat{x}(t)$ and covariance $P(t)$.

Most observers are described recursively as a dynamical system whose input is the measured variables $\begin{bmatrix} u \\ y \end{bmatrix}$ and whose output is the state estimate \hat{x} such as

$$\dot{z} = \hat{f}(t, z, u, y) \quad (18)$$

$$\hat{x} = \hat{h}(t, z, u, y) \quad (19)$$

If the state of the observer, z , is of the same dimension as the state of the system, then it is called a full order observer; if it is of greater dimension then it is called an expanded order observer, and if it is of lesser dimension, then it is called a reduced order observer. For example, the prototype autonomous linear observer Eqs. (10) - (12) can be written as

$$\dot{z} = (A - LC)z + [B - LD \quad L] \begin{bmatrix} u \\ y \end{bmatrix} \quad (20)$$

$$\hat{x} = z \quad (21)$$

$$z(0) = \hat{x}^0 \quad (22)$$

and hence is a full order observer. The state of extended Kalman filter discussed as follows is the pair $z = (\hat{x}, P)$, so it is an expanded order observer. We briefly discuss the Luenberger observer, a reduced order observer for a linear autonomous system in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad (23)$$

$$y = x_1 + Du \quad (24)$$

$$x(0) = x^0 \quad (25)$$

The reduced order observer is given by

$$\dot{z} = (A_{22} - LA_{12})z + [(A_{21} - LA_{11}) + (A_{22} - LA_{12})L](y - Du) \quad (26)$$

$$\hat{x}_1 = y - Du \quad (27)$$

$$\hat{x}_2 = z + L(y - Du) \quad (28)$$

where L is a design parameter. If the model is exact then $\hat{x}_1 = x_1$ and

$$\dot{\tilde{x}}_2 = (A_{22} - LA_{12})\tilde{x}_2 \quad (29)$$

so if the spectrum of the matrix $A_{22} - LA_{12}$ lies in the open left half plane then the error decays to zero exponentially fast. We discuss when L can be so chosen in the next section. For more on reduced order linear observers, (see *Observer Design*).

The state z of the observer is some measure of the likely distribution of the state of the original system given the past observations. If the observer is derived using stochastic arguments, the state of the observer is typically the conditional density of the state of the system given the past observations and the initial information. In the extended Kalman filter, the state $z = (\hat{x}, P)$ is the mean and the covariance of the approximately Gaussian distribution of the true state. For the full and reduced order linear observers described previously, which were derived by nonstochastic arguments, one can view the conditional density as being singular and concentrated at a single point, $\hat{x}(t)$.

2. Observability

The question of whether an observer converges is of paramount importance. A more immediate question is when a nonlinear system Eqs. (15) - (17) admits a convergent observer. This leads to the concepts of observability and detectability which are discussed in (see *Controllability and Observability of Nonlinear Systems*). Briefly two states x^{01}, x^{02} are said to be distinguishable by an input $u(t)$ if the outputs $y^1(t), y^2(t)$ of Eqs. (15) - (17) satisfying the initial conditions $x^0 = x^{01}, x^0 = x^{02}$ differ at some time $t \geq 0$. The system is said to be observable if every pair x^{01}, x^{02} can be distinguished by some input $u(t)$. An input $u(t)$ which distinguishes every pair x^{01}, x^{02} is said to be universal. A system where every input is universal is said to be uniformly observable.

Consider a smooth autonomous nonlinear system without inputs

$$\dot{x} = f(x) \quad (30)$$

$$y = h(x) \quad (31)$$

$$x(0) = x^0 \quad (32)$$

At time $t = 0$ the output and its time derivatives are given by the iterated Lie derivatives

$$y(0) = h(x^0) \quad (33)$$

$$\dot{y}(0) = L_f(h)(x^0) = \frac{\partial h}{\partial x}(x^0) f(x^0) \quad (34)$$

$$\ddot{y}(0) = L_f^2(h)(x^0) = \frac{\partial L_f(h)}{\partial x}(x^0) f(x^0) \quad (35)$$

and so on. If the p -vector-valued functions $h, L_f(h), L_f^2(h), \dots$ distinguish points then clearly the system is observable. For a real analytic system, this is a necessary and sufficient condition for observability. This suggests a way of reconstructing the state of a system, differentiate the output numerous times, and find the state which generates such values. One does not proceed in this fashion because differentiation greatly accentuates the effect of the almost inevitable noise that is present in the observations, and multiple differentiations greatly increase this problem. That is why observers are usually dynamic systems driven by measurements. When such systems are integrated, the effect of the noise is mitigated not enhanced.

For simplicity of exposition, suppose that $n = kp$. If the matrix

$$\begin{bmatrix} \frac{\partial(h)}{\partial x}(x^0) \\ \frac{\partial L_f(h)}{\partial x}(x^0) \\ \vdots \\ \frac{\partial L_f^{k-1}(h)}{\partial x}(x^0) \end{bmatrix} \quad (36)$$

is invertible then the p -vector-valued functions

$$\xi_1 = h(x), \quad (37)$$

$$\xi_2 = L_f(h)(x), \dots, \quad (38)$$

$$\xi_k = L_f^{k-1}(h)(x) \quad (39)$$

are local coordinates around x^0 and in these coordinates the system Eqs. (30) - (32) becomes

$$y = \xi_1 \quad (40)$$

$$\dot{\xi}_1 = \xi_2 \quad (41)$$

$$\begin{aligned} \dot{\xi}_2 &= \xi_3 \\ &\vdots \end{aligned} \quad (42)$$

$$\dot{\xi}_k = \bar{f}_k(\xi) \quad (43)$$

Each ξ_i is a p -vector. Such a system is said to be in observable form, since it is clearly observable. Many algorithms for constructing observers start with the assumption that the system is in observable form. The observable form of a $n = kp$ system with inputs is

$$y = \xi_1 + g_0(\xi, u) \quad (44)$$

$$\dot{\xi}_1 = \xi_2 + g_1(\xi, u) \quad (45)$$

$$\begin{aligned} \dot{\xi}_2 &= \xi_3 + g_2(\xi, u) \\ &\vdots \end{aligned} \quad (46)$$

$$\dot{\xi}_k = \bar{f}_k(\xi) + g_k(\xi, u). \quad (47)$$

where $g_i(\xi, 0) = 0$. Such a system is clearly observable as the input $u(t) = 0$ distinguishes every pair of points, but it may not be uniformly observable. A system

$$y = \xi_1 + g_0(u) \quad (48)$$

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + g_1(\xi_1, u) \\ &\vdots \end{aligned} \quad (49)$$

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} + g_2(\xi_1, \dots, \xi_i, u) \\ &\vdots \end{aligned} \quad (50)$$

$$\dot{\xi}_k = \bar{f}_k(\xi) + g_k(\xi_1, \dots, \xi_k, u). \quad (51)$$

is said to be in uniformly observable form for it is clearly uniformly observable. From the knowledge of $u(t)$, $y(t)$ we can determine $\xi_1(t)$, from the knowledge of $u(t)$, $y(t)$, $\dot{\xi}_1(t)$ we can determine $\xi_2(t)$, etc.

An autonomous linear system is observable if, and only if, the matrix

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (52)$$

is of full column rank in which case C, A is said to be an observable pair. Moreover, for such systems the spectrum of $A - LC$ can be set up arbitrarily to complex conjugation by choice of L . (As a real matrix the spectrum of $A - LC$ is invariant with respect to complex conjugation.) (See *Observer Design*).

A system Eqs. (15) - (17) is detectable, if whenever the outputs are equal $y^1(t) = y^2(t)$ from the initial states x^{01}, x^{02} using the same control $u(t)$, then the state trajectories converge $x^1(t) - x^2(t) \rightarrow 0$.

For an autonomous linear system, the kernel of the matrix Eq. (52) is the largest invariant subspace of the matrix A contained in the kernel of C . It is not hard to show that the system is detectable if, and only if, the spectrum of A restricted to the kernel of Eq. (52) is in the open left half plane. Clearly, the spectrum of $A - LC$ on the kernel of Eq. (52) does not depend on L . The rest of the spectrum of $A - LC$ can be set up arbitrarily to complex conjugation by choice of L .

Hence a linear system admits a convergent observer if, and only if, it is detectable. It is not hard to show that the system Eq. (23) - (25) is detectable if, and only if, the reduced system is.

$$\dot{x}_2 = A_{22}x_2 \quad (53)$$

$$\bar{y} = A_{12}x_2 \quad (54)$$

Hence a linear system admits a convergent reduced order observer if and only if it is detectable.

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Biographical Sketch

Arthur J. Krener was born in Brooklyn, NY on October 8, 1942. He received his BS degree from Holy Cross College in 1964 and his MA and Ph.D. degrees from the University of California, Berkeley in 1967 and 1971, all in Mathematics. Since 1971, he has been at the University of California, Davis where he has been Professor of Mathematics since 1980. He has held visiting positions at Harvard University, the University of Rome, Imperial College of Science and Technology, NASA Ames Research Center, the University of California, Berkeley, the University of Paris IX, the University of Maryland, the University

of Newcastle, Australia and the University of Padua. His research interests are in developing methods for the control and estimation of nonlinear dynamical systems and stochastic processes. Professor Krener is a Fellow of the IEEE. His 1981 IEEE Transactions on Automatic Control paper with Isidori, Gori-Giorgi and Monaco won a Best Paper Award. His 1977 IEEE Transactions on Automatic Control paper with Hermann was recently chosen as one of 25 Seminal Papers in Control in the last century by the IEEE Control Systems Society. He was a Fellow of the John Simon Guggenheim Foundation for 2001-2002.

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