# ANALYSIS AND STABILITY OF FUZZY SYSTEMS

**Ralf Mikut and Georg Bretthauer**
*Forschungszentrum Karlsruhe GmbH, Germany*

**Keywords:** Systems, Linear Systems, Nonlinear Systems, Closed-loop Systems, SISO Systems, MISO systems, MIMO Systems, Fuzzy Systems, Fuzzy Sets, Membership Functions, Fuzzy Operators, Fuzzification, Inference, Defuzzification, Fuzzy Controller, Fuzzy Models, Mamdani-type Fuzzy Systems, Takagi-Sugeno-type Fuzzy Systems, Characteristic Fields, Stability, Lyapunov Functions, Cell-to-cell Mapping, Describing Functions, Hyperstability, Fuzzy-Lyapunov Functions, Performance Measures, Robustness

## Contents





## Summary

This chapter gives an overview of different techniques for the analysis of fuzzy systems. Fuzzy systems are based on sets with gradual membership values and IF-THEN rules. The three procedures of fuzzification, inference, and defuzzification transform numerical input quantities into numerical output quantities. In most cases, the input-output relation is equivalent to a nonlinear function. The presented analysis techniques focus on fuzzy controllers in closed-loop systems. Basic problems of great practical relevance are stability, performance, and robustness of overall system. After introductory remarks, the technique to transform fuzzy systems into functional descriptions will be shown in the second section for Mamdani-type and Takagi-Sugeno-type fuzzy systems. Stability analysis of fuzzy systems is the central subject of the third

section. It gives an overview of special modifications of classical nonlinear techniques for fuzzy systems and fuzzy-based techniques for an online supervision. Further approaches to the analysis of stand-alone fuzzy systems and more simulation-based techniques for performance and robustness analysis will be presented in the fourth section. The chapter ends with a discussion of open problems, future trends, and a review of references for further reading.

## 1. Introduction

In the early years of application of fuzzy logic controllers, most engineers thought that this kind of control solutions does not require any analysis. They assumed that fuzzy systems contain expert knowledge which guarantees an adequate behavior of the control loop. Nevertheless, this optimistic approach often fails, because expert knowledge cannot be collected completely. In addition, fuzzy systems show a very complex and not always predictable behavior for the engineer.

In non-safety-critical applications, the use of fuzzy controllers without a formal analysis might be tolerable after an intensive testing and tuning of the fuzzy controller with the real plant. But this approach is not acceptable for many industrial solutions, where malfunctions of controllers may cause safety problems or damage the plant. Here, fuzzy control competes with classical approaches which provide many analysis techniques to prove stability, robustness, controllability, and a lot of further characteristics.

Fuzzy systems process linguistic terms and rules to describe uncertainties of signals and relations (in the sense of “possibilities”). Nevertheless, they normally have a deterministic input-output behavior. Most fuzzy systems in practice are equivalent to static nonlinear functions.

Objects of analysis are stand-alone fuzzy systems or closed control loops containing fuzzy controllers. The investigation can be made offline during a design phase or online. An analysis of a stand-alone fuzzy system (e.g. completeness of the rule base or stability) is important to find design errors, but insufficient to investigate the closed-loop system, including a fuzzy controller and a plant. As a consequence, techniques to analyze closed-loop fuzzy systems become necessary.

However, each offline analysis of control loops requires a plant model. This contradicts the classical approach of model-free fuzzy controller design. If a mathematical plant model exists, a functional description of the closed control loop results when the fuzzy controller is transformed into a nonlinear system. Then, the definitions and the analysis methods for the resulting system are very similar to the techniques presented in *Control of Nonlinear Systems*. Typical structures of fuzzy controllers can often be exploited to simplify the analysis. Alternatively, the resulting closed-loop system can be analyzed as a fuzzy system, if a fuzzy model of the plant is given. Here, particular definitions are used in analysis e.g. fuzzy stability. Different kinds of hybrid approaches are possible between these strategies. For instance, fuzzy models of the plant can be generated, but the analysis is made after transformation of fuzzy controller and fuzzy plant model into nonlinear mathematical equations.

An online analysis can be done by means of a further fuzzy system for supervision. This supervisor analyzes the real system by means of fuzzy rules on a successful and sufficient behavior similar to a human process operator and can optionally work without an explicit plant model. Mathematical knowledge for the analysis of fuzzy systems can also be formalized into fuzzy rules and used for fuzzy supervision.

The aims of this chapter are

- to demonstrate approaches to transforming fuzzy systems into a nonlinear mathematical description (Section 2),
- to describe different techniques for stability analysis, including special modifications for fuzzy systems (Section 3),
- to present other analysis tasks and methods (Section 4), and
- to discuss open problems and future trends (Section 5).

## 2. Transformation Approaches

### 2.1. Overview

The following section deals with techniques to transform a fuzzy system into a mathematical description with equivalent input-output behavior. In many practical applications, the transformation is based on the input-output mapping of a fuzzy system, including fuzzification, inference, and defuzzification. The fuzzy system mostly has a MISO (multiple-input-single-output) or MIMO (multiple-input-multiple-output) structure. Additional continuous-time or discrete-time dynamic (and mostly linear) subsystems exist only externally and are not part of the fuzzy system.

From a theoretical point of view, fuzzy systems map a vector of membership functions $\boldsymbol{\mu}_x$ to a vector of membership functions $\boldsymbol{\mu}_y$ with $F:\boldsymbol{\mu}_x \rightarrow \boldsymbol{\mu}_y$ (Fig. 1, left). The membership functions characterize linguistic statements (e.g. about the value of a measured variable), but the mapping $F$ is deterministic. This structure is especially suited for fuzzy systems, where the non-defuzzified output of the fuzzy system is connected to the input of the fuzzy system (see Fig. 1, right). Such system structures are of limited practical importance.

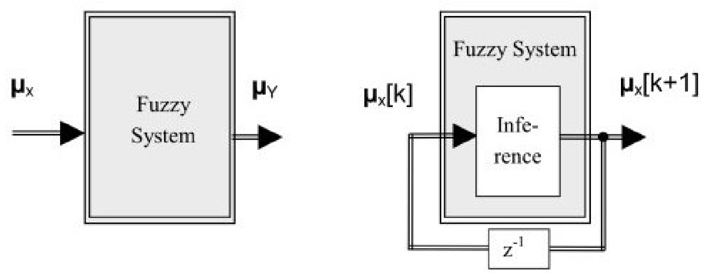


Figure 1: System structure for an analysis of fuzzy systems (left) and example of a simple fuzzy system with external dynamics, where the inference processes rules describing the closed-loop system behavior (right)

In many practical applications, however, the input and output variables of fuzzy systems take on numerical values. As a consequence, the membership functions at the input and output of the fuzzy systems $\boldsymbol{\mu}_x, \boldsymbol{\mu}_y$ in Fig. 1 are singletons. For this, known measured values at the input and a numerical valued decision after a defuzzification at the output of the fuzzy system are assumed. As a result, the fuzzy system maps a vector of numerical input values to a vector of numerical output values. It is equivalent to a deterministic, static, nonlinear system (Fig. 2, left) with

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \tag{1}$$

in case of a Mamdani-type fuzzy system, and

$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \tag{2}$$

in case of a Takagi-Sugeno-type fuzzy system. Here, the additional input variable $\boldsymbol{u}$ characterizes variables which will be processed only in inference and not in fuzzification. It should be noted that different internal structures (e.g. Mamdani-type: linguistic terms as rule conclusions, Takagi-Sugeno-type: functions as rule conclusions) lead to a similar functional, static description.

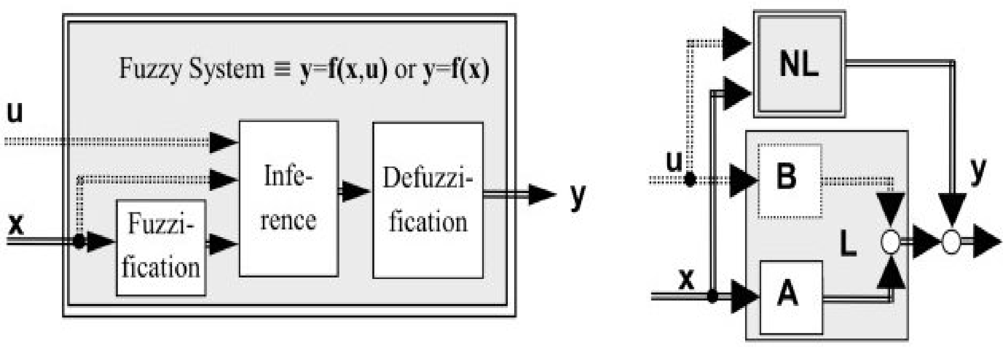


Figure 2: Characteristic schemes for the analysis of Mamdani-type (except for dotted lines and $\boldsymbol{u}$, respectively) or Takagi-Sugeno-type fuzzy systems (including dotted lines and $\boldsymbol{u}$) – as one static nonlinear subsystem (left) or as static linear (L) and additive static nonlinear subsystem (NL) (right)

After an equivalent transformation, the resulting function can also be analyzed as a linear subsystem and a parallel nonlinear subsystem with $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}_{NL}(\boldsymbol{x})$ in case of Mamdani-type systems or $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{f}_{LN}(\boldsymbol{x},\boldsymbol{u})$ in case of Takagi-Sugeno-type systems (Fig. 2, right). This structure is typical for a robustness analysis, where the influence of the remaining nonlinear part will be investigated.

The resulting functions in Eqs. (1-2) are often called “characteristic fields”. They are suitable for different analysis tasks. In the following section, typical characteristic fields for Mamdani- and for Takagi-Sugeno-type fuzzy systems will be discussed.

## 2.2. Mamdani-type Fuzzy Systems

Mamdani-type fuzzy systems with an $s$-dimensional input vector $x$ are defined by rules $R_i, i = 1,\dots,r$ of the type

$$R_i \;:\; \text{IF } (x_1 = A_{1,R_i})\,\text{AND}\cdots\text{AND}(x_s = A_{s,R_i})\text{ THEN } y_k = B_{k,R_i}$$

where $A_{j,R_i}$ is a linguistic term of the *j*-th linguistic input variable in the *i*-th rule and $B_{k,R_i}$ is a linguistic term of the *k*-th linguistic output variable in the *i*-th rule.

The characteristic field of a Mamdani-type fuzzy system depends on the membership functions, the rules, the fuzzy operations for aggregation, activation and accumulation, and the defuzzification method (see *Fuzzy Control Systems*).

To obtain a functional description as in Eq. (1) of the fuzzy system, all these procedures have to be transformed into functions. The resulting function normally has a complicated structure, because many types of membership functions (e.g. triangular, trapezoidal) cause case differentiations.

An often simpler form is given by points with a rule being completely activated and interpolation laws existing between these rules. The interpolation depends crucially on the type of membership functions, the fuzzy operation chosen for disjunctions (OR, T-conorm, $\perp$) and conjunctions (AND, T-norm, $\top$), and the defuzzification method. For some fuzzy operations, the interpolation can be non-monotone.

The function $\boldsymbol{f}$ is mostly continuous, but not differentiable.

The transformation will be demonstrated for an example, a Mamdani-type fuzzy controller with $s = 3$ input (control error $e$, time derivative of control error $\dot{e}$, and reference value $w$, identical membership functions in Fig. 3, left) and the manipulated variable $y$ as output.

A part of the rule base is shown in Table 1 with singletons ($y_{ZE} = 0, y_{PS} = 1, y_{PM} = 2, y_{PB} = 4$, Fig. 3, middle) as conclusions. As defuzzification method, the Center of Singletons method (COS) is used.

| w | ZE | ZE | POS | POS |
|---|---|---|---|---|
| $\dot{e}$<br>e | ZE | POS | ZE | POS |
| ZE | *y*=ZE | *y*=PS | *y*=ZE | *y*=PM |
| POS | *y*=PS | *y*=PM | *y*=PM | *y*=PB |

Table 1: Rule base for the example with ZE Zero, POS Positive, PS Positive Small, PM Positive Medium, PB Positive Big

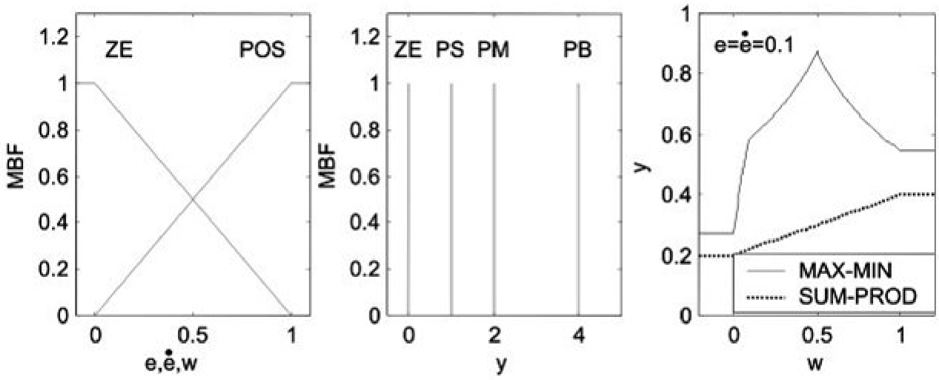


Figure 3: Membership functions (left: input variables $e, \dot{e}$, and $w$, middle: output variable $y$ with singletons $y_{ZE} = 0, y_{PS} = 1, y_{PM} = 2, y_{PB} = 4$) and interpolation behavior as part of a characteristic field (right)

The characteristic field $y = f(e, \dot{e}, w)$ is given by $r = 8$ rules, the input membership functions $\mu_{x_j = A_{j,i}}(x_j)$, the chosen fuzzy operations $\perp$ and $\top$, and the defuzzification method COS:

$$y = \frac{y_{ZE} \cdot \mu_{y=ZE}(e,\dot{e},w) + y_{PS} \cdot \mu_{y=PS}(e,\dot{e},w) + y_{PM} \cdot \mu_{y=PM}(e,\dot{e},w) + y_{PB} \cdot \mu_{y=PB}(e,\dot{e},w)}{\mu_{y=ZE}(e,\dot{e},w) + \mu_{y=PS}(e,\dot{e},w) + \mu_{y=PM}(e,\dot{e},w) + \mu_{y=PB}(e,\dot{e},w)}, \quad (3)$$

with

$$\mu_{y=ZE}(e,\dot{e},w) = \perp(\top(\mu_{e=ZE}(e), \mu_{\dot{e}=ZE}(\dot{e}), \mu_{w=ZE}(w)), \top(\mu_{e=ZE}(e), \mu_{\dot{e}=ZE}(\dot{e}), \mu_{w=POS}(w))),$$
$$\dots$$
$$\mu_{y=PB}(e,\dot{e},w) = \top(\mu_{e=POS}(e), \mu_{\dot{e}=POS}(\dot{e}), \mu_{w=POS}(w))).$$

By using the bounded sum ($\perp$) and the product ($\top$), the characteristic field is described as

$$y(e,\dot{e},w) = (1 + \min(1, \max(w,0))) \cdot (\min(1, \max(e,0)) + \min(1, \max(\dot{e},0))) \quad (4)$$

Here, the minimum and maximum operations result from the trapezoidal membership functions, not from the conjunctions and disjunctions.

Under special conditions, the characteristic field for bounded sum and product is a continuous function with multilinear facets (here: $y = (1 + w) \cdot (e + \dot{e})$ for $0 < e, \dot{e}, w < 1$). Fig. 3 shows the output value of a fuzzy system as a function of $w$ with the constant $e = 0.1, \dot{e} = 0.1$.

For bounded sum ($\perp$) and product $\top$ (SUM-PROD), Eq. (4) yields a piece-wise linear function $y(0.1, 0.1, w) = 0.2 \cdot (1 + \min(1, \max(w,0)))$ (dotted line in Fig. 3, right).

More complicated characteristic fields occur in case of maximum ($\perp$) and minimum ($\top$). The interpolation between $w=0$ and $w=1$ with maximum ($\perp$) and minimum ($\top$) (MAX-MIN) is

$$y(e=0.1,\dot{e}=0.1,w)=\begin{cases} 0.3/1.1 & \text{if } w\le 0 \\ (0.3+4\cdot w)/(1.1+w) & \text{if } 0<w\le 0.1 \\ 0.7/(1.3-w) & \text{if } 0.1<w\le 0.5 \\ 0.7/(0.3+w) & \text{if } 0.5<w\le 0.9 \\ (1.6-w)/(2.1-w) & \text{if } 0.9<w\le 1 \\ 0.6/1.1 & \text{if } w>1 \end{cases} \quad (5)$$

(solid line in Fig.3, right). The non-monotone interpolation in Eq. (5) is dominated by three effects:

- the activation of the rule with $y=PB$ between $0<w<0.1$,
- the activation of two rules with $y=PS$ between $0.9<w<1$, and
- the reduced accumulation of $\mu_{y=ZE}$ caused by the decreasing maximum activation of the two rules with $y=ZE$ between $0.1<w<0.9$ ($\mu_{y=ZE}=0.9$ for $w=0.1,\ldots,\mu_{y=ZE}=0.5$ for $w=0.5,\ldots,\mu_{y=ZE}$ $=0.9$ for $w=0.9$).

Here, the unexpected local maximum at $w=0.5$ can cause different problems in the control loop, as the values of $w$ for constant $e,\dot{e}$ corresponds to a local control gain at the operating point described by $w$. These problems may extend up to instability.

-
-
-



**Bibliography**

Aracil, J.; Gordillo, F. (eds.) (2000). *Stability Issues in Fuzzy Control*, vol. 44 of *Studies in Fuzziness and Soft Computing*. Heidelberg: Physica. [Collection with different approaches for stability analysis of fuzzy systems]

Bandemer, H.; Hartmann, S. (1998). A Fuzzy Approach to Stability of Fuzzy controller. *Fuzzy Sets and Systems* 96, pp. 161-172. [Definition and methods for linguistic stability analysis]

Bretthauer, G.; Opitz, H.-P. (1994). Stability of Fuzzy Systems – A Survey. In: *Proc. 2nd Europ. Congr. on Intelligent Techniques and Soft Computing EUFIT'94*, pp. 283-290. Aachen. [Overview about different approaches for stability analysis of fuzzy systems]

Farinwata, S.S.; Fivel, D.P.; Langari, R. (eds.) (2000). *Fuzzy Control: Synthesis and Analysis.* Wiley. [Collection with different approaches for a mathematical analysis of closed-loop fuzzy systems, including approaches for Lyapunov functions, hyperstability and describing functions]

Johansen, T.A. (1994). Fuzzy Model Based Control: Stability, Robustness and Performance Issues. *IEEE Trans. Fuzzy Systems* 2, pp. 221-233. [Hyperstability analysis and robustness for fuzzy systems]

Kang, H. (1993). Stability and Control of Fuzzy Dynamic Systems via Cell-State Transitions in Fuzzy Hypercubes. *IEEE Trans. Fuzzy Systems* 1, pp. 267-279. [Definition and methods for linguistic stability analysis]

Kiendl, H.; Rüger, J.J. (1995). Stability Analysis of Fuzzy Control Systems Using Facet Functions. *Fuzzy Sets and Systems 70,* pp. 275-286. [Special approach with multilinear Lyapunov functions for fuzzy systems]

Mikut, R.; Bretthauer, G. (1996). A New Approach for Testing the Stability of Multi-Level Fuzzy Systems. In: *Proceedings of the World Automation Congress (WAC'96)*, vol. 5, pp. 495-500. TSI Press Series. [On-line supervision of fuzzy systems and linguistic stability definition]

Tanaka, K.; Sugeno, M. (1992). Stability Analysis and Design of Fuzzy Control Systems. *Fuzzy Sets and Systems* 45, pp. 135-156. [Special approach with Lyapunov functions for Takagi-Sugeno-type fuzzy systems]

Wang, L.-X. (1994). *Adaptive Fuzzy Systems and Control.* Englewood Cliffs, NJ: Prentice Hall. [On-line supervisory control structures for stabilization]

**Biographical Sketches**

**Ralf Mikut** obtained the Dipl.-Ing. degree in Automatic Control at the University of Technology, Dresden in 1994 and the Ph.D. degree at the University of Karlsruhe in 1999. He is now Project Manager at the Institute of Applied Computer Science of the Forschungszentrum Karlsruhe (Karlsruhe Research Center). His research interests are in the areas of fuzzy systems, computational intelligence, medical diagnosis support systems, automatic control, and robotics. He is elected chairman of the Technical Committee 5.22 "Fuzzy Control" of the German Society of Measurement and Automatic Control (GMA).

**Georg Bretthauer** obtained the Dipl.-Ing., Dr.-Ing., and Dr.-Ing. habil. degrees in Automatic Control at the University of Technology, Dresden in 1970, 1977, and 1983, respectively. He is now Professor of Applied Computer Science and Automatic Control at the University of Karlsruhe and the Head of the Institute of Applied Computer Science at the Karlsruhe Research Center. His research interests are in the fields of computational intelligence, knowledge-based systems, and automatic control.

He is elected Chairman of the German Society of Measurement and Automatic Control. He works as a technical expert of German research in the field of automatic control and is member of the administrative council of the European Union of Control associations.