DIFFERENTIAL AND INTEGRAL CALCULUS

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Contents

1. Historical survey

As the name indicates, differential and integral calculus is a combination of integral calculus and differential calculus. However, these two types of calculus have different histories. Integral calculus has two completely different aspects, namely integration, that is merely the "inverse of differential operation" and "integral as a limit of a sum" a concept completely independent of differentiation. The integral as a limit of a sum has a much longer history than differential calculus. For example, Archimedes calculated the areas and volumes of various shapes and solids by inference, even with the exactness of modern mathematics. In comparison, the history of differential calculus is much shorter and its origins can be traced back to the findings of Newton and Leibnitz in the 17th century. Differentiation deals with the instantaneous change of a function, for example, the rate of change of displacement of a moving mass, i.e., velocity. Differentiation is a very valuable finding by itself. However, mathematicians in the 17th century were astonished by the fact that the "inverse of differentiation" simplified calculations even for problems requiring great time and labor if calculated by the "integral as a limit of a sum". There is no doubt that the concept of "instantaneous rate of change" has some risks if considered from an entirely philosophical point of view. However once the concept had been implemented, the inverse of differentiation induced an explosive development in mathematics in terms of practical effectiveness. Since then, by sacrificing logical exactness to some extent, the "inverse of differentiation" view for integration has maintained a central role in the flow of mathematics. A closer look at the history of integration along this flow reveals that integration was handled as "inverse of differentiation" for calculational purposes rather than as a concept based on subdivision approach.
As mathematics began to deal with more abstract objects, problems that could not be solved by intuition began to appear. In the middle of the 19th century, differential and integral calculus was reconstructed on the basis of a deep discussion on the set of real numbers and the so-called $\epsilon-\delta$ description. Concerning integration, as functions to be integrated became more complicated people's priority in interest shifted from calculations to the precise formalism of integration. Integration formalism developed through Riemann's formulation abstracted from the subdivision approach into, in the end, the Lebesgue integral.

This chapter outlines only the elementary aspects of differential and integral calculus from the viewpoint of modern mathematics. First, the basic properties of real numbers and the $\epsilon-\delta$ description of a limit are briefly explained. They are followed by a summary of the convergence and limit of a sequence of numbers, and summaries of continuity, differentiation, and Taylor's formula. As for integration, Riemann's definition is introduced (the Lebsegue integral is explained in) and then integration is discussed in terms of the inverse of differentiation. Based on this discussion, explicit formulae of integrals are summarized for specific types of functions. Finally, the discussion on differential and integral calculus is further extended to functions of many variables.

2. Convergence of Sequences

2.2 Definition of Convergence

A sequence $\{a_n\}$ of numbers $a_n, n = 1, 2, 3, \ldots,$ converges to a number a if a_n goes arbitrarily close to a as n becomes larger. This definition of convergence gives an intuitively clear image of convergence and so could serve as a "definition" of convergence. In fact it had been used as its definition till the middle of the 19th Century and is still being used in introductory or applications-oriented textbooks of calculus. On the other hand logically rigorous definition goes as follows;

A sequence $\{a_n\}$ of numbers *converges to a number* a if for any positive number $\epsilon > 0$ there exists an n_ϵ (depending on ϵ) such that

$$|a_n - a| < \epsilon, \quad \text{for all } n \geq n_\epsilon .$$

If a sequence $\{a_n\}$ converges, then the number a it converges to is unique, and is called the *limit* of the sequence. We denote

$$a = \lim_{n\to\infty} a_n .$$

Thanks to this definition of the limit, many properties can be rigorously proved. For instance, if $\{a_n\}$, $\{b_n\}$ converge to a, b respectively, then

- $\lim_{n\to\infty}\left(ka_n \pm \ell b_n\right) = ka \pm \ell b,$ where $k, \ell \in \mathbf{R},$ (the linearity of limit)
- $\lim_{n\to\infty}\left(a_n b_n\right) = ab,$ and
- if $b \neq 0$ and $b_n \neq 0$ for all n then $\lim_{n\to\infty} a_n / b_n = a/b$.

For exploring calculus, however, one more deep insight into real numbers is required, that is:

2.2. The Basic Property of Real Numbers.

A sequence $\{a_n\}$ converges to some real number if the following two conditions are satisfied.

- Increasing, that is $a_1 \le a_2 \le \cdots \le a_n \le \cdots$, and
- there exists a number M such that $a_n \le M$, for all n.

An *upper bound* of a subset A of real numbers is a real number m such that $a \le m$ for all $a \in A$. A subset A is *bounded from above* if there exists an upper bound of A. The *supremum* of a subset A, denoted by sup A is an upper bound α of A such that no number smaller than α is an upper bound of A. The infimum inf A is defined similarly with the inequality reversed.

From the basic property of real numbers it follows that sup A exists if A is bounded from above. Conversely if one assumes the existence of sup A for any subset A bounded from above, then the basic property above follows. Namely, the existence of the supremum for every upper bounded subset of real numbers is equivalent to the basic property.

A sequence $\{a_n\}$ is called a *Cauchy sequence* if for any positive number ϵ there exists an n_ϵ such that

$$|a_n - a_m| < \epsilon, \quad \text{for all } n, m \text{ greater than } n_\epsilon.$$

The above basic property of real numbers is also equivalent to the property that a sequence of numbers converges if and only it is a Cauchy sequence.

2.3. Real Line

The real numbers can be represented by points on a straight line, where the point representing the number 0 is called the origin and length of the line segment between two points, representing numbers x and y respectively, is supposed to be $|x - y|$. Due to this geometrical representation of real numbers, a real number is often referred to as a point. This pictorial description is useful for a description of a function by a graph. (See Section 3.3.)

3. Continuous Functions

3.2 Continuous Functions and Their Limits

A function f assigns a real number $f(x)$ for each number x belonging to some set D of real numbers, called the domain of the function f. The number $y = f(x)$ is the value of the function f at the point x. In the relation $y = f(x)$ where x varies over the domain D and correspondingly y varies over the set E of real numbers $f(x)$, $x \in D$ x, is called the independent variable and y is called the dependent variable . E is called the range of the function f **.**

Intuitively, continuity of a function $y = f(x)$ means that the dependent variable y varies without jump; thus a "small" change in x implies only a "small" change in the dependent variable y. However the relation between the margins of precision of two "small" changes one in x and the other in y, should be made logically clear and precise. A precise definition goes as follows.

A function $f(x)$ is *continuous* at a point x_0 of its domain if for any positive ϵ there exists a positive number δ such that

$$\left|f(x) - f(x_0)\right| < \epsilon$$

for all x in the domain satisfying $\left|x - x_0\right| < \delta$.

A function is *continuous* if it is continuous at all points in the domain. If a function f is continuous at a_0 and a sequence $\{a_n\}$ converges to a_0, then the sequence $\left\{f(a_n)\right\}$ converges to $f(a_0)$.

The continuity of a function is also formulated in terms of the *limit* of the function; y_0 is the limit of the function $y = f(x)$ at x_0, denoted by

$$\lim_{x \to x_0} f(x) = y_0,$$

if for any positive number ϵ there exists a positive number δ such that

$$\left|f(x) - y_0\right| < \epsilon$$

for all x in the domain satisfying $0 < \left|x - x_0\right| < \delta.$. Then $y = f(x)$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

It follows that the limit of a function can also be described in terms of limits of sequences: $\lim_{x \to x_0} f(x) = y_0$ if and only if $\lim_{n \to \infty} f(a_n) = y_0$ for every sequence $\{a_n\}$ with the limit x_0 and satisfying $a_n \neq x_0$ for all n. Hence limits of sums, products, and quotients of functions follow the same rules as for sequences, for instance

$$\lim_{x \to x_0} \left(f(x) + g(x)\right)$$
$$= \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

3.2 Properties of Continuous Functions.

3.2.1 The Intermediate Value Theorem

Let $f(x)$ be a continuous function defined on an interval. Suppose that a and b are any two points of the interval and that η is any intermediate number between $f(a)$ and $f(b)$. Then the *intermediate value theorem asserts* that there exists at least one point ξ between a and b for which $f(\xi)=\eta$.

3.2.2. Maxima and Minima of Continuous Functions

A function $f(x)$ has a *maximum* over a set X (contained in its domain) at a point c belonging to X if the value $f(x)$ is not exceeded by the value of $f(x)$ at any point x in X; that is $f(c) \geq f(x)$ for all x in X. Similarly, $f(x)$ has a minimum over X at c if $f(c) \leq f(x)$ for all x in X. A function $f(x)$ has an *extremum* at a point c if it has either a *maximum* or *minimum* there. It is known that if $f(x)$ is continuous on an interval $[a,b]$ then $f(x)$ has a maximum over that interval at least at one point in the interval and a minimum over that interval and a minimum over that interval at least at one point in the interval.

A function $f(x)$ has a *relative maximum* at a point c if $f(x)$ has a maximum at c but in this case one compares $f(c)$ just with the value at points sufficiently close to c; that is, if there exists an interval $\left[c_1,c_2\right]$ satisfying $c_1 < c < c_2$ such that $f(c) \geq f(x)$ for all x in it. A *relative minimum* and *relative extremum* are defined similarly.

3.3. The graph of a function

On a plane, a *Cartesian rectangular coordinate system* is defined by two mutually perpendicular real lines called x axis and y axis. Each axis represents real numbers (see the explanation of a real line in §2.3.), the intersection point with the other axis being the origin. Furthermore, the positive directions of the x and y axes point right and up, respectively. Given a point *P* on the plane suppose that the line parallel to the y axis (resp. x axis) passing through *P* crosses the x axis (resp. y axis) at a (resp. b), then the ordered pair (a,b) is called the *coordinates* of the point *P*. The point of intersection of axes is called the *origin* of the plane, and its coordinates are (0, 0).

A plane with a Cartesian rectangular coordinate system is called a *coordinate plane*. Each point in a coordinate plane is indicated by its coordinates.

Given a coordinate plane and a function $y = f(x)$ defined on a domain D, the *graph* of the function f is the set of points $(x, f(x)), x \in D$ in the coordinate plane. For instance, the graph of the function $y = ax + b$ with a, b constants, is a straight line which crosses the y axis (resp. x axis) at $(0, b)$ (resp.$(-b/a, 0)$) if $a \neq 0$, and the graph of the function $y = ax^2 + bx + c$ is a parabola.

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Bibliography

Courant R. and John F. (1989). Introduction to Calculus and Analysis I, II, Springer-Verlag. [A standard text on calculus.]

Marsden J. and Weinstein A. (1980). Calculus I ~ III, Springer-Verlag. [A text on calculus which contains many solved problems and extensive exercises.]

Dieudonne J. (1969). Foundation of Modern Analysis, Academic press. [A book of analysis written in modern "coordinate free style". Differential calculus is treated on Banach spaces.]

Whittaker E.T. (1996). A course of Modern Analysis, 4th ed. Cambridge Mathematical Library, Series. [A classical text on analysis.]

Courant R. and Hilbert D. (1953). Methods of Mathematical Physics ‡ T, Interscience Publ., New York. [An advanced book of Analysis.]

Zwillinger D. (1992). Handbook of Integration , A K Peters. [A handbook of integration formulae.]

Jahnke E. and Emde F. (1945). Tables of functions with formulas and curves, Dover publ., New York. [A handbook of calculus.]

Hairer E. and Wanner G. (1995). Analysis by its History, Undergraduate Texts in Mathematics, Springer-Verlag. [A book on the history of analysis.]

Biographical Sketch

Yoshio Togawa was born on January 5, 1953. He received BS, MS and DS degrees respectively in 1975, 1977, and 1981, all from Waseda University. Since 1977 Dr. Togawa has been with Tokyo Science University where he is professor since April 1992.