OPERATOR THEORY AND OPERATOR ALGEBRA

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Summary

A Hilbert space is a Banach space whose norm comes from an inner product, and it is the most natural infinite-dimensional generalization of the Euclidean space. Operators on a Hilbert space appear in many places, and may be viewed as matrices of infinite size. One can add and multiply them, and furthermore the *-operation (extending the notion of adjoint matrix) can be introduced due to the presence of an inner product. Operator

theory studies individual operators while the theory of operator algebras deals with *-algebras of operators. C^*-algebras and von Neumann algebras are particularly important classes of such *-algebras. In this chapter, some selected topics on operator theory, C^*-algebras, and von Neumann algebras are explained.

1. Hilbert Space

A pre-Hilbert space means a linear space $\mathcal{H}$ (usually over $\mathbf{C}$) equipped with an inner product $\langle \cdot, \cdot \rangle$ satisfying

(i) $(\xi,\eta) \in \mathcal{H}\times\mathcal{H} \to \langle \xi,\eta \rangle \in \mathbf{C}$ is linear in the first variable ξ and conjugate linear in the second variable η,

(ii) $\langle \xi,\xi \rangle \geq 0$ and $\langle \xi,\xi \rangle = 0$ iff $\xi = 0$.

By virtue of (ii) we can set $\|\xi\| = \sqrt{\langle \xi,\xi \rangle}$ (≥ 0), the norm of ξ. The Cauchy-Schwarz inequality $|\langle \xi,\eta \rangle| \leq \|\xi\|\|\eta\|$ guarantees that $\|\cdot\|$ is indeed a norm on $\mathcal{H}$:

(i) $\|\lambda\xi\| = |\lambda|\|\xi\|$ for $\lambda \in \mathbf{C}$,

(ii) $\|\xi+\eta\| \leq \|\xi\| + \|\eta\|$ (Triangle inequality),

(iii) $\|\xi\| = 0$ iff $\xi = 0$.

With this norm a pre-Hilbert space $\mathcal{H}$ becomes a normed linear space so that $d(\xi,\eta) = \|\xi-\eta\|$ defines a metric: a sequence $\{\xi\}_{n=1,2}$ converges to ξ when $\lim_{n\to\infty} d(\xi_n,\xi) = 0$. When this metric is complete (every Cauchy sequence is convergent), $\mathcal{H}$ is called a Hilbert space. In other words, $\mathcal{H}$ is a Hilbert space when $\mathcal{H}$ equipped with the associated norm $\|\cdot\|$ is a Banach space. The norm $\|\cdot\|$ (from $\langle \cdot, \cdot \rangle$ as above) satisfies

$$\|\xi+\eta\|^2 + \|\xi-\eta\|^2 = 2\left(\|\xi\|^2 + \|\eta\|^2\right) \quad \text{(Parallelogram law)}$$

$$\langle \xi,\eta \rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|\xi + i^k\eta\|^2 \quad \text{(Polarization identity)}$$

with $i = \sqrt{-1}$. The parallelogram law is of fundamental importance in handling Hilbert spaces, and it actually characterizes Hilbert spaces. In fact, when the norm $\|\cdot\|$ of a Banach space $\mathcal{H}$ satisfies the parallelogram law, the right side of the polarization identity gives rise to an inner product whose associated norm is $\|\cdot\|$.

The n-dimensional vectors $\mathbf{C}^n$ (with the ordinary vector operations) form a Hilbert space with the inner product

$$\langle \xi,\eta \rangle = \sum_{k=1}^{n} \xi_k \overline{\eta_k} \text{ for } \xi = (\xi_1,\xi_2,\ldots,\xi_n),\ \eta = (\eta_1,\eta_2,\ldots,\eta_n) \in \mathbf{C}^n .$$

The associated norm is $\|\xi\| = \sqrt{\sum_{k=1}^{n} |\xi_k|^2}$, the Euclidean distance. More interesting (infinite-dimensional) examples are as follows:

(i) $\mathcal{H} = L^2(\mathbf{R}; dx)$, which is the space of measurable functions $f(x)$ on $\mathbf{R}$ satisfying the square integrability condition

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty,$$

where dx is the Lebesgue measure on the real line $\mathbf{R}$ and functions are being identified if they differ only on a null set. The linear structure here is given by the point-wise sum of functions, and $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$ gives rise to an inner product.

(ii) Any measure space gives rise to a Hilbert space as in (i). One important special case is $\mathbf{N}$ equipped with the counting measure:
$\ell^2 = \{$ the space of sequences $\xi = \{\xi_k\}_{k=1,2\ldots}$ satisfying $\sum_{k=1}^{\infty} |\xi_k|^2 < \infty\}$ with the inner product $\langle \xi, \eta \rangle = \sum_{k=1}^{\infty} \xi_k \overline{\eta_k}$.

(iii) Let H^2 be the set of analytic functions f on the open disk $\mathbf{D} = \{z \in \mathbf{C}; |z| < 1\}$ satisfying

$$\|f\|_2 = \lim_{r \nearrow 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right)^{1/2} < \infty.$$

This space can be naturally identified as a closed subspace of the L^2-space $L^2(\mathbf{T}; d\theta/2\pi)$ over the torus $\mathbf{T} = \partial\mathbf{D}$ (as will be explained shortly). Hence, H^2 itself is a Hilbert space, known as the Hardy space.

A family $\{e_\alpha\}_{\alpha \in \Lambda}$ in $\mathcal{H}$ is called an orthogonal system when $\|e_\alpha\| = 1$ and $\langle e_\alpha, e_\beta \rangle = 0$ for $\alpha \neq \beta$. Furthermore, when linear combinations of e_α 's form a dense subspace, it is called an orthogonal basis. In many practical situations Hilbert spaces are separable (i.e., possessing a dense sequence) so that separability will be assumed in the rest of our discussion. Then, the above index set Λ is (at most) countable, and hence an orthogonal basis is actually a sequence $\{e_n\}_{n=1,2,\ldots}$ (or a finite sequence $\{e_n\}_{n=1,2,\ldots,m}$ when $\dim \mathcal{H} < \infty$). With such a basis an arbitrary element $\xi \in \mathcal{H}$ can be expressed as $\sum_{k=1}^{\infty} \langle \xi, e_k \rangle e_k$ $(= \lim_{n\to\infty} \sum_{k=1}^{n} \langle \xi, e_k \rangle e_k)$ with $\{\langle \xi, e_k \rangle\}_{k=1,2,\ldots} \in \ell^2$. Consequently, infinite-dimensional (separable) Hilbert spaces are all isomorphic to ℓ^2, and in particular the above $L^2(\mathbf{R}; dx)$ is also isomorphic to ℓ^2 as a Hilbert space.

Let $\{e_n\}_{n \in \mathbf{Z}}$ $(\subseteq L^2(\mathbf{T}; d\theta/2\pi))$ be the (exponential) orthonormal basis defined by $e_n(e^{it}) = e^{\mathrm{int}}$ $(e^{it} \in \mathbf{T})$. The expression $f = \sum_{n \in \mathbf{Z}} a_n e_n$ for $f \in L^2(\mathbf{T}; d\theta/2\pi)$ with $a_n = \langle f, e_n \rangle$ is nothing but the Fourier series expansion. Then the above Hardy space

H^2 can be identified with the closed subspace generated by $\{e_n\}_{n=0,1,2,...}$. More precisely, for $f \in H^2$ radial limits $F(e^{it}) = \lim_{r\nearrow 1} f(re^{it})$ exist almost everywhere (a.e.) on the torus **T**. Then the "radial limit" function $F(e^{it})$ sits in $L^2(\mathbf{T}; d\theta/2\pi)$ and indeed falls into the above mentioned closed subspace, i.e., $F(e^{it}) = \sum_{k=0}^{\infty} a_n e_n$ with $\|f\|_2 = (\sum_{n=0}^{\infty} |a|^2)^{1/2}$. The analytic function $f \in H^2$ is recovered from F by making use of the Poisson integral, and also the power series expansion $f(z) = \sum_{k=0}^{\infty} a_n z^n$ is valid on **D**.

A linear functional $\varphi : \mathcal{H} \to \mathbf{C}$ is called bounded if $\|\varphi\| = \sup\{|\varphi(\xi)|; \xi \in \mathcal{H}_1\} < \infty$, where $\mathcal{H}_1 = \{\xi \in \mathcal{H}; \|\xi\| \le 1\}$ is the unit ball. Let $\mathcal{H}^*$ (the dual space of $\mathcal{H}$) be the set of all bounded linear functionals on $\mathcal{H}$. With the obvious linear structure and the above norm $\|\cdot\|$, the dual space $\mathcal{H}^*$ is a Banach space. For a general Banach space the fact that the dual $\mathcal{H}^*$ contains an abundance of elements is not so obvious and in fact it follows form the Hahn-Banach theorem. However, for the Hilbert space case it is easy. Indeed any $\eta \in \mathcal{H}$ induces the bounded linear functional $\varphi_n : \xi \to \langle \xi, \eta \rangle \in \mathbf{C}$ with $\|\varphi_\eta\| = \|\eta\|$. The Riesz theorem asserts that every element in the dual $\mathcal{H}^*$ arises in this way, which plays an important role in many places.

2. Bounded Linear Operator

A linear operator on a Hilbert space $\mathcal{H}$ means a linear map $T : \mathcal{H} \to \mathcal{H}$. A linear operator T is continuous iff it is bounded, i.e. $\|T\| = \sup_{\xi \in \mathcal{H}_1} \|T\xi\| < \infty$. Many natural linear operators (such as differential operators on function spaces) are unbounded, and very beautiful and useful theories on unbounded operators are known. However, we will mainly deal with bounded linear operators, and they will be simply called operators in what follows. The quantity $\|T\|$ is referred to as the operator norm of *T*. It is indeed a norm on the set $B(\mathcal{H})$ of all (bounded linear) operators with the obvious linear structure, and $B(\mathcal{H})$ is a Banach space. For operators $T, S \in B(\mathcal{H})$ we have $\|TS\| \le \|T\|\|S\|$ so that $B(\mathcal{H})$ is actually a Banach algebra with the unit 1, the identity operator. Linear operators on a Banach space can be also considered, but the theory of operators on a Hilbert space is much richer than the theory of those on a Banach space. What makes the former so is the adjoint operation. Namely, for $T \in B(\mathcal{H})$ there is a unique operator *S* (denoted by T^*, the adjoint of *T*) satisfying

$\langle T\xi, \eta \rangle = \langle \xi, S\eta \rangle$ for each $\xi, \eta \in \mathcal{H}$

(by virtue of the Riesz theorem), which makes $B(\mathcal{H})$ a $*$-Banach algebra (i.e., a Banach algebra with an involution * satisfying $\|T^*\| = |T\|$) .

When $\dim \mathcal{H} = n < \infty$, $B(\mathcal{H})$ is the space of $n \times n$-matrices. Hence an operator can be regarded as a "matrix of infinite size", and one naturally tries to generalize what is known for matrices to the operator setting (although such generalizations are impossible in many cases at least in a straight-forward fashion). Besides the norm topology induced by $\|\cdot\|$, the strong operator topology and weak operator topology are often considered and useful. They are the linear topologies determined by the following families of semi-norms:

$\{p_\xi(\cdot)\}_{\xi\in\mathcal{H}}$ with $p_\xi(T) = \|T\xi\|$ (strong operator topology),

$\{p_{\xi,\eta}(\cdot)\}_{\xi,\eta\in\mathcal{H}}$ with $p_{\xi,\eta}(T) = |\langle T\xi,\eta\rangle|$ (weak operator topology).

2.1. Compact Operator

An operator $T \in B(\mathcal{H})$ is called a compact operator if $T\mathcal{H}_1$ is relatively compact in the norm topology ($T\mathcal{H}_1$ is automatically closed when $\mathcal{H}$ is a Hilbert space). The set $\mathfrak{K}(= \mathfrak{K}(\mathcal{H}))$ of all compact operators is a norm closed two-sided ideal (actually the only non-trivial one) in $B(\mathcal{H})$. For compact operators some results for matrices can be generalized in the most straight-forward fashion. For example, as a generalization of diagonalization by a unitary matrix, one can show the following: if a compact operator T is normal (i.e., $TT^* = T^*T$), then one can find an orthogonal basis $\{e_k\}_{k=1,2,\ldots}$ such that $Te_k = \lambda_k e_k$ for some $\lambda_k \in \mathbf{C}$, that is, T is unitarily equivalent to an infinite diagonal matrix. Furthermore, when λ_k 's are arranged in such a way that $|\lambda_k|$'s are in decreasing order, then we must have $\lim_{k\to\infty} |\lambda_k| = 0$.

2.2. Miscellaneous Operators

(i) An operator T is called self-adjoint if $T^* = T$, that is, $\langle T\xi,\xi\rangle \in \mathbf{R}$ for $\xi \in \mathcal{H}$. A self-adjoint operator S is called positive ($S \geq 0$) if $\langle S\xi,\xi\rangle \geq 0$ for $\xi \in \mathcal{H}$. For self-adjoint operators S_1, S_2 we consider the order $S_1 \geq S_2$ defined by $S_1 - S_2 \geq 0$. For an operator S the following conditions are equivalent: (a) S is positive, (b) $S = A^*A$ with some $A \in B(\mathcal{H})$, (c) $S = B^2$ with a positive operator B. In the last condition a positive operator B is uniquely determined, and denoted by $\sqrt{S}$ (the square root of S).

(ii) For a closed subspace $\mathcal{K}$ in $\mathcal{H}$, $\mathcal{K}^\perp = \{\eta \in \mathcal{H};\ \langle \eta,\xi\rangle = 0$ for each $\xi \in \mathcal{K}\}$ is called the orthogonal complement of $\mathcal{K}$. Any element $\xi \in \mathcal{H}$ can be written uniquely as $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathcal{K}$ and $\xi_2 \in \mathcal{K}^\perp$. The operator $P\ (= P_{\mathcal{K}})\ :\ \xi \in \mathcal{H} \to \xi_1$ satisfies $P^2 = P = P^*$. Such an operator is called a projection. Every projection P arises in this way with $\mathcal{K} = P\mathcal{H}$ so that the projections and the closed subspaces are in bijective correspondence.

(iii) An operator U is called a partial isometry if U^*U is a projection (or equivalently UU^* is a projection). The projections U^*U , UU^* are called the initial and final projections of U. The name partial isometry is justified by the following properties: (a) U sends $U^*U\mathcal{H}$ onto isometrically, (b) U acts as the zero operator in the orthogonal complement $(U^*U\mathcal{H})^{\perp}$.

(iv) When $U^*U = UU^* = 1$, U is called a unitary operator. On the other hand, when $S^*S = 1$, it is called an isometry. Note that $S^*S = 1$ does not imply $SS^* = 1$ (unless $\dim \mathcal{H} < \infty$). A typical example is the unilateral shift operator $S_0 : S_0 e_k = e_{k+1}$ with an orthonormal basis $\{e_k\}_{k=0,1,\ldots}$. In fact, $S_0 S_0^*$ is the projection onto the closed subspace (of co-dimension one) generated by $\{e_k\}_{k=1,2,\cdots}$

2.3. Polar Decomposition and Spectral Decomposition

For an operator T we can find a partial isometry U and a positive operator S satisfying $T = US$ in such a way that the support projection of S (i.e, the smallest projection P satisfying $PS = S$) coincides with the initial projection U^*U . The above partial isometry U and positive operator T are unique subject to the conditions mentioned so far. This decomposition is called the polar decomposition of T. The positive operator S (usually denoted by |T|) is actually the positive square root $\sqrt{T^*T}$.

For a self-adjoint operator T we can find a family $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ of projections satisfying

(i) $\lambda \in \mathbf{R} \to E_\lambda$ is increasing, $E_\lambda \le E_\mu$ if $\lambda \le \mu$

(ii) $E_\lambda = 0 \left(\lambda < -\|T\|\right)$ and $E_\lambda = 1\left(\lambda \ge \|T\|\right)$,

(iii) $\lambda \in \mathbf{R} \to E_\lambda$ is continuous from the right in the strong operator topology,

(iv) $T = \int_{-\infty}^{\infty} \lambda d\mathrm{E}_\lambda$.

From (i) and (iii) the function $\lambda \to \langle E_\lambda \xi, \xi\rangle = \|E_\lambda \xi\|^2$ is increasing and continuous from the right for each $\xi \in \mathcal{H}$, and (iv) (spectral decomposition of T) means

$$\langle T\xi, \xi\rangle = \int_{-\infty}^{\infty} \lambda d\left\|E_\lambda \xi\right\|^2 \qquad \text{(for each } \xi \in \mathcal{H}\text{)},$$

where the right side is understood in the Stieltjes sense. The family $\{E_\lambda\}_{\lambda\in\mathbf{R}}$ is called a spectral family. This technique enables us to perform functional calculus $f(T)$ of T for a bounded Borel function f on $\mathbf{R}$ via $\langle f(T)\xi, \xi\rangle = \int_{-\infty}^{\infty} f(\lambda) d\left\|E_\lambda \xi\right\|^2$. For unbounded self-adjoint operators spectral families are also available, however the condition (ii) is replaced by the more general condition: $\lim_{\lambda\to-\infty} E_\lambda = 0$ and $\lim_{\lambda\to\infty} E_\lambda = 1$ in the strong operator topology.

2.4. Spectrum

A complex number λ is called a resolvent of $T \in B(\mathcal{H})$ when the operator $\lambda 1 - T$ is invertible in $B(\mathcal{H})$. The complement of the set of all resolvents is denoted by $\sigma(T)$, the spectrum of *T*. Note that the $\mathcal{H}$ is finite-dimensional $\sigma(T)$ is simply the set of all eigenvalues of a matrix *T*.

However, the behavior of $\sigma(T)$ $(\subseteq \mathbf{C})$ for an infinite dimensional operator is much more subtle. Nevertheless, the spectrum serves as a fundamental tool for handling operators. The spectrum $\sigma(T)$ is known to be a non-empty closed set sitting in the closed disk of radius $\|T\|$.

On the other hand, the spectrum $\sigma_e(T)$ defined similarly at the level of the quotient C^*-algebra $B(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ (see Section 4.1) is referred to as the essential spectrum of *T*. The classical Weyl-von Neumann theorem states that self-adjoint operators T_1, T_2 are unitarily equivalent modulo the compact operators $\mathfrak{K}$ (*i.e.*, $T_1 - UT_2U^* \in \mathfrak{K}$ for some unitary operator *U*) iff $\sigma_e(T_1) = \sigma_e(T_2)$.

The same characterization remains valid for normal operators (Berg and Sikonia independently, 1971).
$UU^*\mathcal{H}$

3. Operator Theory

Operator theory studies individual operators, and it is very diverse. Some selected topics are briefly outlined in the following.

3.1. Dilation Theory

Let $T \in B(\mathcal{H})$ be a contraction (i.e., $\|T\| \le 1$). Sz. Nagy (1960) showed that one can find a Hilbert space $\mathcal{L}$ containing $\mathcal{H}$ and a unitary operator *U* on $\mathcal{L}$ such that

$$T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}} \qquad \text{(for each } n = 0,1,2,\ldots.)$$

with the projection $P_{\mathcal{H}} \in B(\mathcal{L})$ onto $\mathcal{H}$. Furthermore, the above unitary *U* (called a unitary dilation of *T*) is unique under a certain minimality condition. Various results on dilations are known, and such a dilation technique is useful in operator theory.

For example, the following inequality (originally due to von Neumann, 1951) is an immediate consequence of the preceding result: For a contraction *T* and a polynomial $p(x)$ we have

$$\|p(T)\| \le \max\{|p(\lambda)|;\ |\lambda| \le 1\}.$$

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Biographical Sketch

Hideki Kosaki was born in 1950. Received the degree of Ph D in June, 1980 from University of California, Los Angeles. Dr. Kosaki specializes in functional analysis, especially the theory of operator algebras. Dr. Kosaki has been working on non-commutative integration theory and index theory for von Neumann algebras and is also interested in operator theory, and in fact has investigated various operator inequalities. Dr. Kosaki held various positions: Centre de Physique Théorique, CNRS, Marseille; University of Kansas; Purdue University; University of Pennsylvania; Tulane University; MSRI, Berkeley. Currently Dr. Kosaki is Professor, Graduate School of Mathematics, Kyushu University since1994