INTRODUCTION TO MATHEMATICAL ASPECTS OF QUANTUM CHAOS

Dieter Mayer
University of Clausthal, Clausthal-Zellerfeld, Germany

Keywords: arithmetic quantum chaos, Anosov system, Berry-Tabor conjecture, Bohigas-Giannoni-Schmit conjecture, classically chaotic system, classically integrable system, geodesic flow, Hecke operator, Hecke cusp eigenfunction, holomorphic modular form, invariant tori, Kolmogorov-Sinai entropy, Laplace-Beltrami operator, L-functions, Maass wave form, microlocal lift, quantization of Hamiltonian flow, quantization of torus maps, quantum chaos, quantum ergodicity, quantum limit, quantum unique ergodicity, random matrix theory, Rudnick-Sarnak conjecture, semiclassical limit, semiclassical measure, sensitivity in initial conditions, spectral statistics,Wignerdistribution,zetafunctions

Contents

Summary

This is a review of rigorous results obtained up to now in the theory of quantum chaos and also of the basic methods used thereby. This theory started from several conjectures about the way how the behavior of a quantum system is influenced by its classical limit being integrable or chaotic. Numerical calculations suggested that Wigner's statistical approach to the spectra of heavy nuclei via large random matrices seems to be applicable quite generally to quantum systems with chaotic classical limit.

On the other hand, the construction of semiclassical solutions of the stationary Schrödinger equation for quantum systems with integrable classical limit, whose generic orbits cover invariant tori in phase space, lead to the expectation, that eigenstates of classically chaotic quantum systems should, like their generic classical orbits on the energy hypersurface, be equidistributed in configuration space.

Even if progress in understanding general quantum systems rigorously is slow, there is a class of systems where important results have been obtained, namely those systems, which have nice arithmetic properties and are directly related to problems in number theory. For them the methods developed there can be applied. On the other hand, this connection influenced also research in number theory, as for instance on the statistical

properties of zeros of number theoretic functions like zeta or L-functions. This whole circle of problems is now combined under the name “arithmetic quantum chaos”, and the results there constitute the main body of this report.

Since the whole theory is still in plain development with many open problems and therefore far from being a complete theory, this chapter tries only to review and describe the current situation and has certainly to be reworked in the future.

1. Introduction

1.1. Einstein’s Quantization Rules

The origin of the theory of quantum chaos, which in the physics literature is sometimes also called ‘quantum chaology’ (Berry, 1987), is a paper by Einstein (1917), which at his time did not find much attention in the scientific community. In this paper he discussed the so called ‘old quantum mechanics’ of Bohr, Sommerfeld and Epstein and their approach to pass from classical mechanics to the quantum spectrum of a Hamiltonian system with N degrees of freedom and Hamilton function $H = H(\underline{x})$. He gave a coordinate independent formulation of Sommerfeld’s quantization rules which are valid for general completely integrable systems and not only for the separable ones among them, which can be reduced to N uncoupled systems with one degree of freedom. For such a completely integrable system with N degrees of freedom Einstein’s quantization conditions have the form

$$\oint_{\gamma_i} \underline{p}d\underline{q} = n_i h, \quad n_i \in \mathbb{N}, \; 1 \le i \le N, \tag{1}$$

with h Planck’s constant. The γ_i’s denote “closed curves in $\underline{q}$-space to which all closed curves can be reduced by continuous deformations” to use Einstein’s original formulation. In modern language, they are the N cycles determining a basis of the fundamental group of an invariant N-torus $\mathbb{T}_N$ in the phase space $\Gamma = \{\underline{x} \in T^*(M)\}$, the cotangent bundle of configuration space M of the completely integrable system. In local canonical coordinates the so called microstates $\underline{x}$ are then given by $\underline{x} = (\underline{q}, \underline{p}) \in \mathbb{R}^{2N}$. The main remark Einstein's in (1917) however concerns the fact, that for systems not integrable, like for instance Poincarés 3-body system, this method does not work.

Einstein therefore formulates the problem, how to determine the quantum spectrum of a system whose classical limit in the extreme case has no invariant tori at all and whose generic trajectory in phase space is dense on the entire energy shell $\Gamma_E = \{\underline{x} \in \Gamma : H(\underline{x}) = E\}$. This is the case for what one calls nowadays classically chaotic systems and whose time evolution $\Phi_t^H : \Gamma_E \to \Gamma_E$ on the energy hypersurface Γ_E depends sensitively on initial conditions. This means, that the Hamiltonian flow

Φ_t^H has at least one positive Liapunov exponent $\lambda = \lim_{t\to\infty} \frac{\log((D\Phi_t^H)_{\underline{x}}(\underline{v}))}{t} > 0$ for almost all $\underline{x} \in \Gamma_E$ with respect to the invariant measure on Γ_E induced from Liouville measure $d\mu_{\mathrm{L}}(\underline{x}) = d\underline{q}\, d\underline{p}$, and some tangent vector $\underline{v} \in T_{\underline{x}}(\Gamma_E)$. On the other hand, the time evolution of a quantum system with Hamilton operator $\hat{H}_\hbar = -\frac{\hbar^2}{2m}\Delta + V(\underline{q})$, with Δ the Laplace operator in $\mathbb{R}^N$, which is given by the unitary operator $U_\hbar(t) = \exp\left[-(\frac{i}{\hbar}\hat{H}_\hbar t)\right]$ in a Hilbert space $\mathcal{H}$, cannot be chaotic in this sense for $\hbar \neq 0$, since it is in general quasiperiodic and hence does not depend sensitively on the initial state in $\mathcal{H}$. This already shows the singular character of the so called semiclassical limit $\hbar \to 0$ of quantum physics which obviously is not a small perturbation of the limit $\hbar = 0$ of classical mechanics.

1.2. The Berry-Tabor Conjecture on the Local Statistics of the Eigenvalues of Classically Integrable Systems

Numerous numerical calculations have shown that in the semiclassical limit $\hbar \to 0$ one can find nevertheless typical fingerprints of its classical limit in the spectrum of a quantum system, which depend on this classical limit being integrable or chaotic. Already from Bohr's correspondence principle one expects physically, that a quantum system with Hamilton operator $\hat{H}_\hbar = -\frac{\hbar^2}{2m}\Delta + V(\underline{q})$ should behave more or less classically when Planck's constant $\hbar$ is "small". A typical case for instance for this is the situation, when the system's de Broglie wavelength $\lambda = \frac{h}{\sqrt{2m(E-V(\underline{q}))}}$ is very small compared to the characteristic distance over which the potential V varies appreciably, the so called short wave length respectively high frequency limit.

Another example is the case of a free particle, moving in a bounded region like a billiard table, with Hamilton operator $\hat{H}_\hbar = -\frac{\hbar^2}{2m}\Delta$ with $\Delta = \partial_x^2 + \partial_y^2$ the Euclidean Laplace operator with vanishing boundary conditions, where the semiclassical limit $\hbar \to 0$ corresponds to the high energy behavior $E \to \infty$ of the quantum system.

The numerical investigations of different quantum systems resulted in a couple of conjectures both concerning properties of the eigenvalues $\lambda_i(\hbar)$ and the eigenstates $\psi_i(\hbar)$ of the Hamilton operator, given by $\hat{H}_\hbar \psi_i(\hbar) = \lambda_i(\hbar)\psi_i(\hbar)$, which indeed are the main focus of the research in quantum chaos over the last years: for the eigenvalues $\lambda_i(\hbar)$ of a quantum system whose classical limit is completely integrable, Berry and Tabor (1977) formulated the conjecture, that generically the local statistics of its appropriately rescaled eigenvalues should be Poissonian in the semiclassical limit $\hbar \to 0$. This means for instance for the consecutive level spacing distribution $P_n(s) = \frac{1}{n}\sum_{i=1}^{n} \delta(s - \hat{\lambda}_{i+1} + \hat{\lambda}_i)$ of the unfolded eigenvalues $\hat{\lambda}_i$ with unit mean spacing distance, that $\lim_{n\to\infty} P_n(s) = P(s)$ should be given by $P(s) = e^{-s}$, analogous to the waiting times between consecutive completely independent random events.

1.3. The Bohigas, Giannoni and Schmit Conjecture for the Local Statistics of the Eigenvalues of Classically Chaotic Systems

For quantum systems with chaotic classical limit on the other hand, Bohigas et al (1984) suggested in analogy to Wigner's approach to the spectra of heavy nuclei, whose classical dynamics is expected to be highly chaotic, that their rescaled spectra should follow in the semiclassical limit $\hbar \to 0$ the spectral statistics of the eigenvalues of certain Gaussian ensembles of large matrices depending on the systems invariance properties under time reversal (respectively under space rotations for systems with half-integer spins). The corresponding Gaussian measures have the form $d\mu_N(\hat{H}) = C_n \exp(-\frac{\mathrm{trace}\hat{H}^2}{v^2})d\hat{H},\ d\hat{H} = dH_1 \cdots dH_f$, (where f denotes the number of independent real variables of the matrix $\hat{H}$) and are supported on large real symmetric $N \times N$ matrices for systems invariant under time reversal respectively on large $N \times N$ Hermitian matrices for systems lacking time reversal invariance as for instance systems with magnetic fields. The measure $d\mu_N(\hat{H})$ on symmetric matrices is obviously invariant under conjugation of $\hat{H}$ by orthogonal matrices, hence its name Gaussian orthogonal ensemble (GOE), whereas the measure on the Hermitian matrices is invariant under conjugation by unitary matrices, hence its name Gaussian unitary ensemble (GUE). For the common distribution $P(\lambda_1, \cdots, \lambda_N)$ of the eigenvalues $\{\lambda_i\}$ one then finds $P(\lambda_1, \cdots, \lambda_N) = C_{N,\beta} e^{-\sum_{i=1}^{N}\left(\frac{\lambda_i^2}{4v^2}\right)} \prod_{i<j} |\lambda_i - \lambda_j|^{\beta}$, where $\beta = 1$ for the GOE and $\beta = 2$ for the GUE ensemble. From this one derives for large N for the consecutive level spacing of the unfolded eigenvalues to a close approximation for the GOE ensemble the so called Wigner surmise $P(s) \sim \frac{\pi}{2} s \exp(-\pi s^2/4)$, respectively for the GUE ensemble $P(s) \sim \frac{32s^2}{\pi^2}\exp(-4s^2/\pi)$. Hence for chaotic systems consecutive levels of the quantum system should repel each other, whereas in the integrable limit case they should accumulate. For the level density

$$\rho_{N,\beta}(\lambda) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_{N,\beta}(\lambda, \lambda_2, \cdots, \lambda_N) d\lambda_2 \cdots d\lambda_N$$

one gets in the large N limit independent from β

$$\begin{aligned} \rho_N(\lambda) &\sim \frac{1}{2\pi}\frac{1}{Nv^2}\sqrt{(4Nv^2 - \lambda^2)} && \text{for } |\lambda| < 2\sqrt{Nv^2} \\ &\sim 0 && \text{for } |\lambda| > 2\sqrt{Nv^2} \end{aligned}$$

.

For the distribution of the variable $x = \frac{\lambda}{\sqrt{Nv^2}}$ hence one gets Wigner's semicircle law

$$\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2} \quad \text{for} \quad |x|<2$$

$$= 0 \quad \text{for} \quad |x|>2 .$$

Rigorous results concerning these conjectures on the local spectral statistics of quantum systems in the semiclassical limit have been obtained up to now only for systems with nice arithmetic properties, to which methods from number theory can be applied.

1.4. Berry's Conjecture on the Equidistribution of Eigenfunctions for Classically Chaotic Systems

Another characteristic of a quantum system where possible fingerprints of its classical behavior can be found, is the morphology of its bound states in the semiclassical limit, that means the eigenfunctions of its Schrödinger operator $\hat{H}_\hbar = -\frac{\hbar^2}{2}\Delta + V(x)$, when $\hbar$ tends to zero.

1.4.1. Arnold's Quasimodes

The reason for such an expectation can be understood when looking at systems with completely integrable classical limit. One knows from semiclassical quantum mechanics, which started for systems with one degree of freedom with the so called JWKB approximation, named after the mathematician Jeffreys and the physicists Wentzel, Kramers and Brillouin, that one can use the invariant tori of these systems to construct the so called "quasimodes" of Arnold (1972). In special cases these quasimodes turn out to be indeed approximate solutions of the stationary Schrödinger equation (Jakobson and Zelditch, 1999) with eigenvalues determined by Einstein's quantization conditions, which were later corrected by the above authors respectively Keller and especially Maslov. These corrected conditions can be shown to follow basically from single-valuedness of these quasimodes ψ which in local coordinates have the following form

$$\psi(\underline{q}) = \sum_l A_l(\underline{q}) \exp\left(\frac{i}{\hbar} S_l(\underline{q})\right), \qquad (2)$$

with $S_l(\underline{q}) = \int_{\underline{q}_0}^{\underline{q}} \underline{p}_l(\underline{q}')d\underline{q}'$, where $\underline{p}_l(\underline{q})$ is a preimage of $\underline{q}$ under the projection $\pi_N : \mathbb{T}^N \to \mathbb{R}^N$

and the sum is over all these preimages. This can be applied for instance to a particle of mass $m=1$ moving in a potential $V = V(\underline{q})$ in $\mathbb{R}^N$ with completely integrable

Hamilton function $H(\underline{p},\underline{q})=\frac{p^2}{2}+V(\underline{q})$, whose Hamilton operator $\hat{H}_\hbar=-\frac{\hbar^2}{2}\Delta+V(\underline{q})$ defines a self-adjoint linear operator in the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^N)$. It turns out that the amplitude $A_l(\underline{q})$ and therefore the function ψ has singularities at the so called caustic points, which are the images of the singular points of the projection π_N (Berry, 1983). For instance in the case $N=1$ and the Hamilton function $H(q,p)=\frac{p^2}{2m}+V(q)$ one finds for the amplitude $A(q)\sim|E-V(q)|^{-\frac{1}{4}}$. To get solutions of the Schrödinger equation without singularities, according to Maslov's theory, which generalizes the so called connecting formulas of the JWKB solution in dimension $N=1$ at the classical turning points $E=V(q)$, one has to construct also analogous functions $\psi_{\underline{p}}$ in momentum space $\underline{p}$ or some mixed N-dimensional $(q_{i_1},\cdots,q_{i_l},p_{i_{l+1}},\cdots,p_{i_N})$- space, whose inverse Fourier transforms determine smooth functions at the caustics. Remains only to glue all these functions together at the caustics in the phase space (Lazutkin,1993). This gluing process and single valuedness lead finally to corrected quantization conditions of the form (Lazutkin,1993).

$$\oint_{\gamma_i}\underline{p}d\underline{q}=(n_i+\frac{m_i}{4})h,\quad n_i\in\mathbb{N},\quad 1\le i\le N\ , \tag{3}$$

where the integers m_i, the Maslov indices of the closed paths γ_i, are determined by the way γ_i passes through the singular set of the projection map $\pi_N:\mathbb{T}^N\to\mathbb{R}^N$. To get this way solutions of the Schrödinger equation at least up to order $\hbar^2$ the amplitude functions A_l have to fulfill so called transport equations, which determine them uniquely up to a multiplicative constant (Lazutkin,1993). Obviously these quantization conditions determine via the paths $\gamma_i, 1\le i\le N$ an invariant torus $\mathbb{T}^N_{\underline{n}}$ and hence also an energy value $E_{\underline{n}}$ of the classical Hamilton function H with $H_{|\mathbb{T}^N_{\underline{n}}}=E_{\underline{n}}$. It is known, that these values are indeed approximate eigenvalues of the Schrödinger operator (Lazutkin, 1993), whereas in general only linear combinations of the above quasimodes ψ, constructed via the invariant tori, determine approximate eigenfunctions (Lazutkin,1993; Jakobson and Zelditch, 1999). According to the KAM Theorem of Kolmogorov, Arnold and Moser invariant tori exist also for weak perturbations of completely integrable systems, which can be used to determine this way at least part of the spectrum of the Schrödinger operator for such systems (Lazutkin,1993). By construction, the quasimodes “concentrate" in a certain sense to be explained later, around these invariant tori $\mathbb{T}^N_{\underline{n}}$ in the energy shell $\Gamma_{E_{\underline{n}}}$ of the classical phase space.

1.4.2. The Weyl Quantization and Wigner's Distribution

In a classically chaotic system, as for instance an ergodic Hamiltonian system, on the other hand a generic orbit covers densely the whole energy shell Γ_E. This led Berry

(1983) to his general semiclassical eigenfunction hypothesis. To formulate this hypothesis, consider the phase space distribution $W_\psi(\underline{q},\underline{p})$ introduced in 1932 by E. Wigner for any quantum state ψ in the Hilbert space $L^2(\mathbb{R}^N)$ of a system with N degrees of freedom, and defined as

$$W_\psi(\underline{q},\underline{p}) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N} d\underline{x}\exp-(i\frac{\underline{p}\cdot\underline{x}}{\hbar})\psi^*(\underline{q}-\frac{\underline{x}}{2})\psi(\underline{q}+\frac{\underline{x}}{2}). \qquad (4)$$

Then, according to Berry's eigenfunction hypothesis, each semiclassical eigenstate $\psi_\hbar$ for $\hbar \to 0$ should have a Wigner distribution $W_{\psi_\hbar}(\underline{q},\underline{p})$ which is concentrated on the region of phase space explored over infinite time by a typical orbit of the corresponding classical system, that means, on an invariant torus for a classically completely integrable system, respectively on the entire energy shell for a classically ergodic system. In the latter case, the Wigner distribution of a normalized eigenstate $\psi_\hbar = \psi_\hbar(\underline{q})$ then has the form

$$W_{\psi_\hbar}(\underline{q},\underline{p}) \approx \frac{\delta(E - H(\underline{q},\underline{p}))}{\int_\Gamma d\underline{q}d\underline{p}\delta(E - H(\underline{q},\underline{p}))}.$$

Furthermore he suggested, that the individual semiclassical eigenstates $\psi_\hbar$ of classically ergodic systems should behave like Gaussian random fields, that means random functions of several variables, whose finite dimensional distributions

$$F_{\underline{q}_1,\cdots,\underline{q}_k}(x_1,\cdots,x_k) = \mathrm{Prob}\{\psi(\underline{q}_1) \le x_1,\cdots,\psi(\underline{q}_k) \le x_k\}$$

are multivariate Gaussian functions for all $\underline{q}_1,\cdots,\underline{q}_k$ and arbitrary $k \in \mathbb{N}$ (Berry, 1977). Such Gaussian random fields can be constructed for instance by superposing a large number of plane waves with uniformly distributed random phases. This random wave model has been checked experimentally for certain microwave resonators (Stöckmann, 2006)

Even if semiclassical quantum mechanics has a rigorous foundation in the theory of pseudo-differential operators and Fourier integral operators (Zworski, 2012), there are few rigorous results concerning the above formulated conjectures for general quantum systems with either completely integrable or chaotic classical limit.

1.4.3. Trace Formulas

There do not exist many methods which allow one to relate classical and quantum physics: one of them are trace formulae like Selberg's trace formula (Selberg, 1954) or Gutzwiller's semiclassical trace formula (Gutzwiller, 1991; Combescure et al, 1999), which connect the periodic orbits of a classical chaotic Hamiltonian flow with the trace of its Hamilton operator. To get the flavor of such trace formulas, consider a particle moving freely on a compact surface M of constant negative curvature, which we will discuss in more detail in Section 3.1. The corresponding chaotic Hamiltonian flow is the geodesic flow on this surface, whereas its Hamilton operator in units, where $-\frac{\hbar^2}{2m}=1$, is the hyperbolic Laplace-Beltrami operator $\Delta := \Delta_{hyp} = -y^{-2}(\partial_x^2+\partial_y^2)$. If $0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\to\infty$ denote its eigenvalues and l_γ is the length of a prime periodic orbit γ of the flow, then the Selberg trace formula for this system has the following form (see for instance (Marklof, 2012)):

$$\sum_{j=0}^{\infty} h(\rho_j) = \frac{\text{Area}(M)}{4\pi}\int_{-\infty}^{\infty} h(\rho)\tanh(\pi\rho)\rho d\rho + \sum_{\gamma}\sum_{n=1}^{\infty}\frac{l_\gamma g(nl_\gamma)}{2\sinh(\frac{nl_\gamma}{2})}.$$

Thereby $\rho_j=\sqrt{\lambda_j-\frac{1}{4}}$ and h denotes an even function on the complex z plane analytic in the strip $|\operatorname{Im} z|\leq\sigma, \sigma>\frac{1}{2}$, and decaying fast enough at infinity, so that the infinite sum over the eigenvalues converges, respectively g denotes the Fourier transform of h. Obviously, it is not straightforward to extract from such a trace formula the local statistics of the eigenvalues or the morphology of the eigenfunctions. It is therefore not surprising, that most of the rigorous results in the theory of quantum chaos have been obtained by completely different methods, unknown before to the quantum physics community, which however can be applied only for a special class of systems. These are systems, whose classical phase space and dynamics have nice arithmetic properties and their behavior in the semiclassical limit is closely related to known problems in number theory. A typical example is the above mentioned geodesic flow on a surface of constant negative curvature and its quantization, the Laplace-Beltrami operator. Its spectrum is known to be directly related to the theory of automorphic and modular functions for the Fuchsian group defining the surface (Sarnak, 1997) and well studied in analytic and algebraic number theory. Such systems belong to a special branch of quantum chaos, so called arithmetic quantum chaos. Another class of systems also belonging to this branch, are quantized symplectic maps of symplectic manifolds like the so called cat map of the 2-torus, which we will discuss later.

-

-

-

Bibliography

References

[1] Artin E. (1924), Quadratische Körper im Gebiet der höheren Kongruenzen I, II, *Math. Z.* 19, 153–296 [In this paper Artin introduces among other things his famous zeta function and formulates certain conjectures about this function]

[2] Anantharaman N. (2008), Entropy and the localization of eigenfunctions, *Ann. of Math*. (2) 168, 435–475.[The author proves in this paper a lower bound for the Kolmogorov-Sinai entropy of general semiclassical limit measures]

[3] Anantharaman N. (2010), Eigenfunctions of the laplacian on negatively curved manifolds: a semiclassical approach. *Clay Math. Proceedings*, 10, 389–438. [This is a very readable review article on semiclassical quantum mechanics]

[4] Anantharaman N. and Nonnenmacher S. (2007), Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold. *Ann. Inst. Fourier (Grenoble)* 57, 2465–2523. [The authors prove in this paper that semiclassical measures for Laplace eigenfunctions on Anosov manifolds can have at most half of their mass on invariant classical orbits]

[5] Anantharaman N. and Nonnenmacher S. (2007), Entropy of semiclassical measures of the Walsh-quantized Baker's map, *Ann. Henri Poincare* 8, 37–74. [The authors study in this paper a special quantization of the baker map and an entropy bound for the resulting semiclassical measures]

[6] Anantharaman N. and Zelditch S. (2007), Patterson-Sullivan Distributions and Quantum Ergodicity, *Ann. Henri Poincarè* 8 no. 2, 361–426. [The authors establish in this paper another rigorous relation between classical and quantum physics for the geodesic flow on compact surfaces of constant negative curvature by relating the Patterson-Sullivan - to Helgason's distributions on the boundary of the hyperbolic surface]

[7] Arnold V. (1972), Modes and quasimodes, *Funct. Analysis. and Appl*. 6, 94–101. [Arnold introduces in this short announcement in Russian the quasimodes which can be approximate solutions of the stationary Schrödinger equation]

[8] Berry M. (1977), Regular and irregular semiclassical wavefunctions, *J. Phys. A: Math. Gen*. 10, 2083–2091. [Berry formulates here his conjecture that the eigenfunctions of quantum systems with chaotic classical limit should be Gaussian random waves]

[9] Berry M. (1983), Semi classical mechanics of regular and irregular motion, in "*Chaotic behaviour of deterministic systems* ", Les Houches Lectures XXXVI, 195–211, eds. G. Iooss et al., Elsevier Science [These are the very readable notes oflectures of the author on semiclassical quantum mechanics both for integrable and chaotic classical systems]

[10] Berry M. (1987), Quantum chaology, Bakerian Lecture 1987, *Proc. Roy. Soc. London A* 413, 183–198. [This is a short nice lecture about the content of quantum chaos]

[11] Berry M. and Keating J. (1999), The Riemann Zeros and Eigenvalue Asymptotics, *SIAM Review* 41, No. 2, 236–266 [A very clear report on the development of the analogy between eigenvalue asymptotics in quantum mechanics, classical dynamical chaos and prime number theory]

[12] Berry M. and Keating J. (1998), $H = xp$ and the Riemann Zeros, in "*Supersymmetry and Trace Formulae: Chaos and Disorder*", eds. I. Lerner et al., 355– 367, Plenum Publ. N.Y. [In this paper the authors speculate that the imaginary parts of thezeros of Riemann's zeta function are the eigenvalues of a certain quantization of the classical Hamiltonian $H(x, p) = xp$]

[13] Berry M. and Tabor M. (1977), Level clustering in the regular spectrum, *Proc. Roy. Soc. London A* 356, 375–394. [The authors formulate in this paper their conjecture about the Poissonian nature of the quantum spectrum of classically integrable systems]

[14] Bleher P. (1999), Trace formula for quantum integrable systems, lattice point problems and small divisors, in "*Emerging Applications of Number Theory*", IMA Vol. Math. Appl. 109, 1–38, Springer Verlag New York. [In this article the eigenvalue statistics of the Laplacian on tori is related to the lattice point problem]

[15] Bohigas O., Giannoni M., Schmit C. (1984), Characterization of chaotic quantum spectra and universality of level fluctuations, *Phys. Rev. Lett*. 52, 1–4. [The authors formulate in this paper their famous conjecture about the eigenvalue statistics of quantum systems with chaotic classical limit]

[16] Bourgain J. and Lindenstrauss E. (2003), Entropy of quantum limits, *Commun. Math. Phys*. 233, 153–171. [The authors prove in this paper a lower bound for the K-S entropy of semiclassical measures]

[17] Bourgain J. (2007), *A remark on quantum ergodicity for cat maps*, Lect. Notes Math., Springer Verlag, Berlin, 1910, 89–98. [The author answers in this paper a question of Kurlberg and Rudnick concerning quantum ergodicity for the eigenfunctions of the quantumpropagator for the cat map]

[18] Bouzouina A. and De Bievre S. (1996), Equipartition of the eigenfunctions of quantized maps on the torus, *Commun. Math. Phys.* 178, 83–105 [In this paper the authors prove Shnirelman's quantum ergodicity theorem for quantized maps of the 2-torus]

[19] Brooks S. (2010), On the entropy of quantum limits for 2-dimensional cat maps, *Commun. Math. Phys*. 293, 231–255. [The author proves that the weight of a semiclassical measure carried by its ergodic components of high entropy must be larger than the one carried by the components of low entropy]

[20] Bunimovich L. (1974), The ergodic properties of certain billiards, *Funct. Analys. Prilosh*. 8 (3), 254–255 [In this paper the famous Bunimovich stadium billiard and its ergodic properties are discussed]

[21] Bunimovich L. (1979), On the Ergodic Properties of Nowhere Dispersing Billiards, *Commun. Math. Phys*. 65, 295-312. [The author proves here ergodicity of several billiard-flows]

[22] Cheng Z. and Lebowitz J. (1991), Statistics of energy levels in integrable quantum systems. *Phys. Rev*. A 44, 3399–3402. [The authors study in this paper numerically a one parameter family of integrable systems on the torus whose local quantum spectrum for generic parameter values tends to be Poissonian whereas its fluctuations seem to be non-Gaussian]

[23] Cheng Z., Lebowitz J., Major P. (1994), On the number of lattice points between two enlarged and randomly shifted copies of an oval, *Probab. Theory Related Fields* 100, 253–268. [The authors determine here the asymptotic behavior of lattice points between two enlarged and shifted regions in the form of an oval]

[24] Colin de Verdiere Y. (1977), Quasi-modes sur les varietes Riemanniennes. *Invent. Math.* 43 15–52. [In this paper the author studies the quasi-modes originally introduced in a short announcement by V. Arnold]

[25] Colin de Verdiere Y. (1985), Ergodicite et fonctions propres du laplacien, *Commun. Math. Phys*. 102, 497–502. [The author extends in this paper a result of S. Zelditch on a Theorem announced by Shnirelmann to general geodesic flows which are ergodic on some Riemannian manifold]

[26] Colin de Verdiere Y. (1998), Une introduction a la mechanique semi-classique. *L' enseignement mathematique* 44, 23–51. [This is a very nice summary of the methods used in semiclassical quantum mechanics]

[27] Combescure M., Ralston J., Robert D. (1999), A proof of Gutzwiller's semiclassical trace formula using coherent states decomposition, *Commun. Math. Phys.* 202 463–480. [The authors prove in this paper that Gutzwiller's trace formula is rigorous in the semi-classical limit]

[28] Degli Esposti M. (1993), Quantization of the orientation preserving automorphisms of the torus, *Ann. Inst. Henri Poincare* 58, 323–341. [In this paper the author gives a rigorous quantization procedure for hyperbolic torus maps]

[29] Degli Esposti M., Graffi S., Isola S. (1995), Classical limit of the quantized hyperbolic toral automorphisms, *Commun. Math. Phys*. 167, 471–507.[The authors prove in this paper quantum unique ergodicity for certain sequences of eigenfunctions of the quantum propagator for torus maps]

[30] Deligne P. (1974), La conjecture de Weil I, *Publ. Math. IHES* 43, 273–307. [In this paper Deligne proves the famous conjectures of A. Weil about the zero's of Artin's zeta function for algebraic varieties]

[31] Egorov Y. (1969), The canonical transformations of pseudo-differential operators. *Uspekhi Mat. Nauk* 24 (5), 235–236. [In this paper Egorov proves his Theorem on the relation between quantum evolution of observables and their classical evolution in the semiclassical limit]

[32] Eichler M. (1955), *Lectures on modular correspondences*, Tata Institute of Fundamental Research (TIFR) 9. [In these lecture notes Eichler gives a general approach to the Hecke operators in the theory of

automorphic forms]

[33] Einsiedler M. (2009), What is measure rigidity? *Notices Americ. Math. Soc*. 56, 600–601. [Explains to non-experts what measure rigidity is]

[34] Einsiedler M. and Lindenstrauss E. (2010), Diagonal actions on locally homogeneous spaces, in '*Homogeneous flows, moduli spaces and arithemetic*', Clay Math. Proceedings (AMS, Providence, R.I.). [Lecture notes with an introduction to the ideas and the proof of Lindenstrauss on quantum unique ergodicity for compact arithmetic surfaces]

[35] Einstein A. (1917), Zum Quantensatz von Sommerfeld und Epstein, *Verhandl. Deutsche Physik. Gesellschaft* 19, 82–92. [This is the famous article of Einstein, where he shows that the usual quantization conditions of the old quantum mechanics do not work for non-integrable systems. It can be considered the beginning of the theory of quantum chaos]

[36] Eskin A., Margulis G., Moses S. (2005), Quadratic forms of sign (2,2) and eigenvalue spacings on rectangular 2-tori, *Ann. of Math*. (2) 161, 679–725. [The authors characterize through Diophantine conditions those flat tori whose spectral two point functions converge in the large eigenvalue limit to the Poissonian ones]

[37] Faure A. and Nonnenmacher S., De Bievre S. (2003), Scarred eigenstates for quantum cat maps of minimal periods, *Commun. Math. Phys*. 239, 449–492. [The authors construct here semiclassical measures for eigenfunctions of the quantized cat map which are partially supported on periodic orbits of the map]

[38] Faure A. and Nonnenmacher S. (2004), On the maximal scarring for quantum cat map eigenstates, *Commun. Math. Phys.* 245, 201–214. [The authors derive in this paper a maximal bound for the mass which a semiclassical measure for the cat map can have on invariant orbits]

[39] Guerra F., Rosen L., Simon B. (1975), The $P(\Phi)_2$ Euclidean quantum field theory as classical statistical mechanics, Ann. of Math. 101 (1975), 111–259. [This is a review article on the relation between relativistic quantum field theory, Euklidean quantum field theory and classical statistical mechanics]

[40] Gutkin B. (2010), Entropic bounds on semiclassical measures for quantized one-dimensional maps. Commun. Math. Phys. 294, 303–342. [The authors proves lower entropy bounds for semiclassical measures for certain quantized maps of the interval]

[41] M. Gutzwiller (1991), The semi-classical quantization of chaotic Hamilton systems, in "*Chaos et Physique Quantique*" (Les Houches 1989) 201–250, North Holland, Amsterdam. [These are lecture notes on the semiclassical quantization of chaotic Hamiltonian systems]

[42] Hassel A. (2010), Ergodic billiards that are not quantum unique ergodic, Ann. of Math. (2) 171, 605–619.[In this paper the author proves rigorously the existence of semiclassical measures supported on families of periodic orbits, so-called bouncing ball trajectories, in the stadium billiards]

[43] Hejhal D. (1976,1983), "The Selberg trace formula for PSL(2,$\mathbb{Z}$)", Vol.1, Lect. Notes Math. 548, Springer Verlag Berlin, Vol.2, Lect. Notes Math. 1001, Springer Verlag Berlin. [These are two standard references for the proof of Selberg's trace formula for cocompact and cofinite Fuchsian groups]

[44] Hannay J. and Berry M. (1980), Quantization of linear maps on a torus-Fresnel diffraction by a periodic grating, *Physica* D 1 (1980), 267–291. [In this paper the authors introduce a quantization procedure for hyperbolic maps of the torus]

[45] Helffer B., Martinez A., Robert D. (1987), Ergodicité et limite semi-classique, *Commun. Math. Phys.* 313, 313–326. [In this paper the authors prove for a Hamilton operator $\hat{H}_\hbar$, determined by an Hamilton function $\mathrm{H}(\underline{x})$, whose flow Φ_t^H is ergodic on the compact energy surface $H(\underline{x}) = E$, that in the semiclassical limit $\hbar\to 0$ almost all the bound states of $\hat{H}_\hbar$ with energy near to E are distributed according to Liouville measure on the surface $H(\underline{x}) = E$]

[46] Helgason S. (1981), *Topics in harmonic analysis on homogeneous spaces*, Progress in Mathematics, 13, Birkhäuser, Boston, Mass. [A standard monograph on harmonic analysis on two dimensional complete manifolds of constant curvature]

[47] Heller E. (1984), Bound state eigenfunctions of classically chaotic Hamiltonian systems: scars of periodic orbits, *Phys. Rev. Lett.* 53, 1515–1518. [The author reports in this paper about his numerical

investigations of the eigenfunctions of the Laplacian on different integrable and chaotic billiard regions and introduces the so called scars for the stadium billiard, which are concentrated around the bouncing ball geodesics]

[48] Holowinsky R. and Soundararajan K. (2010), Mass equidistribution of Hecke eigenforms, *Ann. of Math*. (2) 172, 1517–1528. [The authors prove quantum unique ergodicity for the Hecke eigenfunctions of the Laplacian on the modular surface]

[49] Iwaniec H. and Kowalski E. (2004), *Analytic Number Theory*, AMS, Colloquium Publications, vol 53. [Presumably one of the best monographs on analytic number theory]

[50] Iwaniec H. and Sarnak P. (2000), *Perspectives on the analytic theory of L-functions*, GAFA Special Volume, Part II, 705–741. [This is a review article on several problems related to L-functions]

[51] Jakobson D. (1994), Quantum unique ergodicity for Eisenstein series on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$, *Annales Inst. Fourier* 44 (5), 1477–1504. [In this paper the author extends a result of Luo and Sarnak on equidistribution of the Eisenstein series for the modular group to their Wigner distributions on the cotangent bundle]

[52] Jakobson D. and Zelditch S. (1999), Classical limits of eigenfunctions for some completely integrable systems, in " Emerging Applications of Number Theory ", eds. D. Hejhal et al., *IMA Math and Appl*. vol. 109, 329–354 Springer Verlag, New York [Besides old results on the limits of eigenfunctions the authors show that any measure on the cotangent bundle of the standard n-sphere S^n invariant under both the geodesic flow and the time reversal transformation can be the quantum limit of eigenfunctions of the Laplacian on the sphere. Another result is the independence of quantum limits from quantization procedures]

[53] Katz N. and Sarnak P. (1999a), Zeroes of zeta functions and symmetry, *Bull. Am. Math. So*c. 36 (1), 1–26. [In this review article the authors explain the reasons for universality of the statistical behavior, determined by random matrix theory, of the zero's of families of zeta functions]

[54] Katz N. and Sarnak P. (1999b), "Random Matrices, Frobenius Eigenvalues, and Monodromy " *Am. Math. Soc. Coll. Publ*. vol. 45, Providence, R.I.. [This book contains the details of the authors work on universal statistical behavior of the zero's of families of zeta and L-functions]

[55] Kelmer D. (2010), Arithmetic quantum unique ergodicity for symplectic linear maps of the multidimensional torus, *Ann. of Math*. 171, 815-879. [In this paper the author shows that on higher dimensional tori eigenfunctions of the quantum propagator can localize on invariant submanifolds attached to isotropic rational subspaces]

[56] Kurlberg P. and Rudnick Z. (2000), Hecke theory and equidistribution for the quantization of linear maps of the torus, *Duke Math. J.* 103, 47–77. [The authors construct a family of commuting operators for the quantum propagator in analogy to the Hecke operators in the theory of automorphic functions]

[57] Kurlberg P. and Rudnick Z. (2001), On quantum ergodicity for linear maps of the torus, *Commun. Math. Phys*. 222, 201–227. [It is shown, that there is a density one sequence of integers so that along this sequence all eigenfunctions of the quantum propagator become uniformly distributed]

[58] Kurlberg P. and Rudnick Z. (2005), On the distribution of matrix elements for the quantum cat map, *Ann. of Math.* 161, 489–507.[The authors formulate the conjecture, that the limiting fluctuations of matrix elements of the Hecke eigenfunctions are distributed like certain weighted sums of traces of independent matrices in SU(2) and verify this for the second and fourth moment.]

[59] Lazutkin V. (1993), "*KAM Theory and Semiclassical Approximations to Eigenfunctions*", Ergebnisse der Mathematik und ihrer Grenzgebiete Vol. 24, Springer Verlag, Berlin. [Standard text book on semiclassical quantum mechanics with an appendix of Shnirelmann on the quantization of chaotic systems]

[60] Lewis J. and Zagier D. (2001), Period functions for Maass wave forms I. *Ann. of Math*. (2) 153, 191–258. [Extension of the Eichler-Manin-Shimura cohomology for modular forms to Maass wave forms]

[61] Lindenstrauss E. (2006), Invariant measures and arithmetic quantum unique ergodicity, *Ann. of Math.* (2) 163, 165–219. [Characterization of semiclassical measures and proof of quantum unique ergodicity for compact arithmetic surfaces]

[62] Lindenstrauss E. (2007), Some examples how to use measure classification in number theory. In

‘*Equidistribution in number theory, an introduction*’, NATO Sci. Ser. II Math. Phys. Chem., volume 237, 261–303. (Springer, Dordrecht 2007). [Discusses Ratner’s measure classification theorem, gives an introduction to entropy theory of algebraic group actions and their applications to the quantum unique ergodicity problem]

[63] Luo W. and Sarnak P. (1995), Quantum ergodicity of eigenfunctions on PSL(2,$\mathbb{Z}$)\H_2, *Inst. Hautes Etudes Sci. Publ. Math.* 81, 207-237.[The authors prove the quantum unique ergodicity conjecture for the Eisenstein series of the modular group and show that possibly existing exceptional sequences for the cusp forms must be very sparse]

[64] Luo W. and Sarnak P. (2003), Mass equidistribution for Hecke eigenforms, Comm. *Pure Appl. Math*. 56, 874-891.[The authors establish equidistribution results for holomorphic Hecke eigenforms]

[65] Luo W. and Sarnak P. (2004), Quantum variance for Hecke eigenforms, *Ann. Sci. Ecole Norm. Sup.* (4) 37, 769–799.[The authors calculate the variance of quantum observables introduced by Zelditch and show that it coincides with the classical variance up to a factor given by the value of a L-function]

[66] Margulis G. and Mohammadi A. (2011), Quantitative version of the Oppenheim conjecture for inhomogeneous quadratic forms, *Duke Math. J.* 158, 121–160 [Gives an application of a quantitative version of the Oppenheim conjecture to a twisted Laplacian on flat 2-tori]

[67] Marklof J. (2000a, 2001a), The Berry-Tabor conjecture, Proc. 3rd Europ. Congr. Math., Barcelona 2000, Progress in Math. 202, 421–427. [Report on recent progress in the proof of the Berry-Tabor conjecture]

[68] Marklof J. (2000b,2001b), Level spacings statistics and integrable dynamics, Proc. XII-th Internat. Congr. f Math. Physics, London 2000, 359–363, Internat. Press, Boston. [Conference report on the Berry-Tabor conjecture for a twisted Laplace operator on the flat torus]

[69] Marklof J. (2003), Pair correlation densities of inhomogeneous quadratic forms, *Ann. of Math*. 158, 419–471. [The author proves, that the two point function for the eigenvalues of a certain twisted Laplace operator on the flat torus is Poissonian]

[70] Marklof J. (2006), Arithmetic quantum chaos, *Encycl. of Math. Phys*. Vol. 1, eds. J.-P. Francoise et al., Elsevier Oxford, 212–221. [This is a conference report on the status of arithmetic quantum chaos]

[71] Marklof J. (2012), Selberg’s trace formula: an introduction, in “*Hyperbolic geometry and applications in quantum chaos and cosmology*”, eds. J. Bolte et al., London Math. Soc. Lect. Notes Series 397, 83–119, Cambridge University Press. [A nice elementary introduction to Selberg’s trace formula and some of its applications]

[72] Mayer D. (1991), The thermodynamic formalism approach to Selberg’s zeta function for PSL(2,$\mathbb{Z}$), *Bull. Am. Math. Soc.* 25, 55–60. [It is shown, that Selberg’s zeta function for the modular group can be expressed as the Fredholm determinant of a transfer operator for the geodesic flow on the modular surface]

[73] Major P. (1992), Poisson law for the number of lattics points in a random strip with finite area, *Prob. Theo. Rel. Fields* 92, 423–464. [Gives the proof of a conjecture by Sinai on the distribution of lattice points for a region in the plane with random boundary]

[74] Miyake T. (1989), “*Modular Forms*” Springer Verlag, Berlin. [Classical textbook on the theory of modular forms]

[75] Montgomery H. (1973), The pair correlation of zeroes of the zeta function, Proc. Symp. Pure Math. 24, 181–193.[This contains the proof, that the two-point function of the unfolded zeros of Riemann’s zeta function coincide with the GUE prediction for certain test functions]

[76] Meurmann T. (1990), On the order of the Maass L-function on the critical line. *Colloq. Math. Soc. Janos Bolyai* 51, 325–354. [The author derives here subconvexity bounds for L-functions of degree two on the critical line Re s=1/2]

[77] Murty R. (2004), Lectures on symmetric power L-functions. Lectures at the Fields Institute (semester program on automorphic forms) www. mast. queensu.ca /~murty / [These are lecture notes for graduate students and non experts]

[78] Nonnenmacher S. (2010a), *Notes on the Minicourse* “*Entropy of chaotic eigenstates* ”, CRM Proc.

and Lect. Notes, Vol. 52, 1–41.[Lecture notes on the author's and N. Anantharaman's work on entropy bounds for semiclassical measures in hyperbolic systems]

[79] Nonnenmacher S. (2010b), *Anatomy of quantum chaotic eigenstates*, Seminaire Poincaré XIV 1–43, arxiv:1005.5598 (to appear in Birkhäuser Verlag) [This is a very nice review article on the semiclassical behavior of bound states of quantum systems with chaotic classical limit including certain billiard flows, geodesic flows on compact Riemannian manifolds and symplectic hyperbolic maps of tori]

[80] Nonnenmacher S. and Voros A. (1998), Chaotic eigenfunctions in phase space, *J. Stat. Phys*. 92, 431–518. [This paper gives a very detailed discussion of the semiclassical limit and the distribution of the zeros of eigenstates of quantized maps of the 2-torus]

[81] Odlyzko G.(1981/1982), Correspondence with G. Polya about the origin of the Hilbert-Polya conjecture, www.dtc.umn.edu/ odlyzko/polya/index.html [This is the correspondence of A. Odlyzko with G. Polya]

[82] Odlyzko G. (1987), On the distribution of spacings between zeros of the zeta function, *Math. of Comput*. 48, 273–308. [Report on the author's numerical work on the zeros of Riemann's zeta function]

[83] Odlyzko G. and Hiary G. (2012), The zeta function on the critical line: numerical evidence for moments and random matrix theory models, *Math. Comput*. 81, 1723–1752. [The authors report on similarities of the statistics of the Riemann zeros and the ones for large random matrices]

[84] Riviere G. (2010a), Entropy of semiclassical measures in dimension 2, *Duke Math. J.* 155, 271–335. [Proof of a stronger entropy bound conjectured by Nonnemacher et al. in dimension two]

[85] Riviere G. (2010b), Entropy of semiclassical measures for nonpositively curved surfaces, Ann. Henri Poinc. 11, 1085–1116. [Extension of the strong entropy bound of the author to nonpositively curved surfaces]

[86] Riviere G. (2011), *Entropy of semiclassical measures for symplectic linear maps of the multidimensional torus*, IMRN vol. 2011, Issue 11, 2396–2443.[The author derives a bound for the KS entropy of semiclassical measures for higher dimensional hyperbolic torus maps]

[87] Rudnick Z. and Sarnak P. (1994), The behavior of eigenstates of arithmetic hyperbolic manifolds, *Commun. Math. Phys.* 161, 195–213. [It is shown in this paper, that there is no strong localization of quantum limits on totally geodesicmanifolds for such manifolds and Berry's random wave model for the eigenfunctions does not hold in general]

[88] Rudnick Z. and Sarnak P. (1996), Zeros of principal L-functions and random matrix theory, *Duke Math. J*. 81 (2), 269–322. [It is shown that the *n*-point correlations of the zeros of primitive principal L-functions coincide universally with the GUE predictions]

[89] Rudnick Z. (2005), A central limit theorem for the spectrum of the modular domain, *Ann. Henri Poincare* 6, 863–883. [One of the few rigorous results concerning the conjecture of Bohigas et al. on the spectral statistics of quantum systems with chaotic classical limit]

[90] Rudnick Z. (2005), On the asymptotic distribution of zeros of modular forms, *IMRN*, 2059–2074. [The author discusses in this paper consequences of the QUE conjecture for the distribution of zeros of holomorphic Hecke forms]

[91] Rudnick Z. (2006), Eigenvalue statistics and lattice points, Colloquio De Georgi, Pisa (www.math.ac.il/ rudnick/pub.html) [Lectures at the Scuola Normale de Pisa]

[92] Rudnick Z. (2007), The arithmetic theory of quantum maps, in "*Equidistribution in Number Theory, an Introduction* " eds. A. Granville et al., 331–342, Springer Verlag.[A very nice review article also for non-experts]

[93] Rugh H. (1992), The correlation spectrum for hyperbolic analytic maps. *Nonlinearity* 5 no. 6, 1237–1263 [In this paper the author introduces for two dimensional hyperbolic analytic maps a transfer operator which makes use both of the expanding and contracting properties of the maps]

[94] Ruelle D. (2002), Dynamical zeta functions and transfer operators, Notices Amer. Math. Soc. 49, 887–895. [Review article on dynamical zeta functions and the transfer operator method]

[95] Sarnak P. (1995), Arithmetic quantum chaos, Schur Lectures, Israel Math. Conf. Proc. [Gives a complete review of results and open problems in quantum chaos theory at that time]

[96] Sarnak P. (1997), Values at integers of binary quadratic forms, in "Harmonic Analysis and Number

Theory ", CMS Conf. Proc. 21, 181–203 A.M.S., Providence R.I. [It is shown in this paper, that for random flat tori the limiting two-point correlation function becomes Poissonian]

[97] Sarnak P. (2003), Spectra of hyperbolic surfaces, *Bull. Amer. Math. Soc*. 40, 441–478. [Very nice review article on solved and open problems concerning the spectrum of the Laplace-Beltrami operator on surfaces of constant negative curvature]

[98] Sarnak P. (2011), Recent progress on the quantum unique ergodicity conjecture, *Bull. Am Math. Soc.* 48 (2), 211–228. [Gives a complete description of presently known results in quantum unique ergodicity]

[99] Selberg A. (1954), Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Ind. Math. Soc*. 20, 47–87. [Famous paper, in which Selberg's trace formula and Selberg's zeta functions are introduced]

[100] Shiffmann B. and Zelditch S. (1999), Distribution of zeros of random and quantum chaotic sections of positive line bundles, *Commun. Math. Phys.* 200, 661–683.[The authors study in this paper limit distributions of zeros of certain sequences of holomorphic sections of high powers L^N of a positive holomorphic Hermitian line bundle L over a compact complex manifold M.]

[101] Shnirelman A. (1974), Ergodic properties of eigenfunctions, *Uspehki Mat. Nauk*, 29 (6), 181–182. [Announcement of Shnirelman's Theorem on quantum ergodicity]

[102] Sinai Y. (1991), Poisson distribution in a geometrical problem, *Adv. Sov. Math. AMS Publ*. 3, 199–215. [The author shows, that for generic F in a certain function space the numbers $\lambda = F(\underline{m})$, $\underline{m} \in \mathbb{Z}^2$ are asymptotically Poisson distributed]

[103] Soundararajan K. (2010a), Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash H$, *Ann. of Math*. (2) 172, 1529–1538.[Proof of quantum unique ergodicity for the Hecke eigenstates on the modular surface]

[104] Soundararajan K. (2010b), *Quantum unique ergodicity and number theory*. Lectures at the Arizona Winter School 'Number Theory and Dynamics', The South West Center for Arithmetic Geometry. [These are lecture notes on the proof of the author and Holowinsky of quantum unique ergodicity for holomorphic modular forms for the modular group]

[105] Stöckmann H.-J. (2006), Chaos in microwave resonators, *Seminaire Poincarè* IX, 1–40 [Gives a review of experimental results on microwave billiards]

[106] VanderKam J. (2000), Correlations of eigenvalues on multi-dimensional flat tori, Commun. *Math. Phys.* 210, 203–223. [It is shown, that almost all flat *k*-tori $\mathbb{T}^k$ have Poissonian *n*-correlation functions for all $2 \leq n \leq (k/2)$]

[107] T. Watson (2001), Rankin triple products and quantum chaos, Arxiv: 0810.0425v3 [math.NT], [This is the Ph-D thesis of Watson relating quantum ergodicity to special values of Rankin-Selberg L-functions]

[108] Weil A. (1941), On the Riemann hypothesis in function fields. *Proc. Nat. Acad. Sci. US*A 27, 345–349. [In this short communication A. Weil announces the proof of Artin's conjectures for curves over a finite field]

[109] Wigner E. (1932), On the quantum correction for thermodynamic equilibrium, *Phys. Rev*. 40, 749–759. [In this paper the author introduces the random matrix approach to the spectra of heavy nuclei]

[110] Zelditch S. (1987), Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math*. J. 55, 919–941. [Contains the complete proof of Shnirelman's Theorem for compact hyperbolic surfaces]

[111] Zelditch S. (1992), Selberg trace formulae and equidistribution theorems, Memoirs of AMS 96, No. 465. [In this paper the author extends his quantum ergodicity theorem to non-compact surfaces like the modular surface]

[112] Zelditch S. (2010), Recent developments in mathematical quantum chaos, in "*Current developments in Mathematics 2009*", ed. S. Yau, Intern. Press of Boston. [This review article is mainly devoted to results for the quantum unique ergodicity problem and the author's work on the geometry of nodal sets of eigenfunctions]

[113] Zworski M. (2012), 'Semiclassical Analysis', Graduate Studies in Mathematics 138, *Amer. Math.*

Soc.. [This certainly will become one of the standard textbooks on semiclassical methods]

Biographical Sketch

D. H. Mayer received both his Diploma in Physics and the degree Dr. rer. nat. from the University of Munich, the latter in 1972. He was assistant professor in Theoretical Physics at the Aachen Institute of Technology, and as a research fellow of the German Science Foundation (DFG) a postdoc at the Institute des Hautes Etudes Scientifiques (IHES) at Bures sur Yvette (France) and the Simon Frazer University at Vancouver (Canada). After his Habilitation in Theoretical Physics at the Aachen Institute of Technology in 1979 he became a Heisenberg Fellow of the German Research Foundation with research stays at the IHES and the Mathematics Department of Warwick University. He replaced full professorships in Mathematics at the University of Heidelberg and in Theoretical respectively Mathematical Physics at the Aachen Institute of Technology and at the Universities of Essen and Giessen. After several visiting professorships at the Research Center Juelich, the IHES in Bures sur Yvette and the Max Planck Institute for Mathematics in Bonn he became in 1991 Professor of Theoretical Physics at the University of Clausthal. There he served as Dean of the Physics Department and of the Faculty for Natural Sciences. After his retirement in 2008 he got a Lower Saxony Professorship for Dynamical Systems, Automorphic Spectral Theory and Number Theory at the University of Clausthal. His interests include the thermodynamic formalism in classical statistical mechanics and the theory of dynamical systems, dynamical and number theoretic zeta functions, the transfer operator approach to quantum chaos and automorphic spectral theory respectively the ergodic theory of circle maps with singularities.