STOCHASTIC CALCULUS

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Summary

Consider an integral

$$\int_a^b X(s)dY(s)$$

where $X(\cdot)$ and $Y(\cdot)$ are stochastic processes. The most important case is when $Y(\cdot)$ is a Brownian motion. The definition of this integral was given by Ito, who also described its fundamental properties. This integral is also useful to evaluate the local time of the Brownian motion. Ito also gave a definition of the differential of a non-differentiable Markov process.

1. Stochastic Integral

In the article *Construction of Random Functions and Path Properties* we treated the Brownian motion as a possible model of the motion of a microscopic particle in a liquid or in a gas. This model seems to be correct when the underlying liquid resp. gas is homogeneous and macroscopically motionless (in other words, no external forces are acting on it). However, when this is not the case—for example when the temperature is changing in space and/or in time—the situation is more complicated. A possible mathematical description is the following. Assume that there exists a deterministic (non-random) function $\sigma(t) >$ ($t \geq 0$) which describes the inhomogeneity (in time) of the medium in which the particle moves. Let $X(t)$ be the location of the particle at time t and assume that

$$X(t_2) - X(t_1) = \sigma(t_1)(W(t_2) - W(t_1)) + o(t_2 - t_1)$$

where $W(t)$ is the ordinary Brownian motion and $o(\cdot)$ is Landau's notation for

$\lim_{t\to\propto} t^{-1}o(t) = 0$. Hence the location of the particle at time t is

$$X(t) = \int_0^t \sigma(s)dW(s) + X(0).$$

However, as we mentioned in the article *Construction of Random Functions and Path Properties*, the Brownian motion is nowhere differentiable and even its sample functions are not of bounded variation. Hence the above Stieltjes integral is meaningless in its original sense. Several new definitions were given of the above "stochastic integral."

In many important applications the function $\sigma(t)$ is a stochastic process instead of a deterministic function. Hence we give the definition of the above integral in this more general situation. It is the so-called Ito integral.

On the stochastic process $\{\sigma(t),\ t \geq 0\}$ we assume some regularity conditions. The most important one is the following:

For any $0 < \tau < \propto$ the process $\{\sigma(t),\ 0 \leq t < \tau\}$ is independent from $\{W(t) - W(\tau),\ t \geq \tau\}$; in other words, the state of the medium before τ does not have any influence to the random microscopic motion of the particle after τ. Note that we did not have to assume any independence property for $\sigma(\cdot)$. Hence $\sigma(t)W(t)$ or $\sigma(t)W(t) - \sigma(\tau)W(\tau)$ $(t \geq \tau)$ may depend on $\sigma(t)$ or $\sigma(t)W(t)$ $(t < \tau)$.

The following two conditions are more technical and they can be replaced by weaker ones.

Let $\sigma(t)$ be continuous a.s. and square-integrable a.s. in a given interval $0 \leq a < b < \propto$ and

$$\int_a^b \mathbf{E}\sigma^2(t)dt < \infty.$$

Now we turn to the definition of the Ito integral.

Let $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of the interval (a,b) and consider
$I(a,b) = \sum_{i=0}^{n-1} \sigma(t_i)(W(t_{i+1}) - W(t_i)).$

Then the Ito integral

$$\int_a^b \sigma(t)dW(t)$$

is defined as the limit of $I(a,b)$ as $n \to \propto$ and $\max_{0\leq i\leq n-1}(t_{i+1} - t_i) \to 0$. This definition seems to be very close to the classical Riemann–Stieltjes integral. The most important difference is the following: in case of the Riemann–Stieltjes integral the limit of

$$\sum_{i=0}^{n-1} f(t_i)(g(t_{i+1})-g(t_i))$$

is equal to the limit of

$$\sum_{i=0}^{n-1} f(\xi_i)(g(t_{i+1})-g(t_i)) \qquad (t_i \le \xi_i < t_{i+1})$$

if $\int_a^b fdg$ exists. However, in the case of an Ito integral the limit of $I(a,b)$ and that of

$$\sum_{i=0}^{n-1} \sigma(t_{i+1})(W(t_{i+1})-W(t_i))$$

or that of

$$\sum_{i=0}^{n-1} \sigma\left(\frac{t_i+t_{i+1}}{2}\right)(W(t_{i+1})-W(t_i))$$

might be very different.

Clearly the Ito integral is a linear operator just like the classical integrals. Here we present some further properties:

(a) the process $X(t)=\int_a^t \sigma(s)dW(s)$ is continuous a.s.,

(b) $\mathbf{E}\int_a^b \sigma(s)dW(s)=0,$

(c) $\mathbf{E}\int_a^b \sigma_i(s)dW(s) \int_a^b \sigma_2 dW(s)=\int_a^b \mathbf{E}\sigma_1(s)\sigma_2(s)ds.$

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Biographical Sketch

P. Révész was born in 1934. He gained his Ph.D. in Budapest in 1958 in Budapest. He was Associate Professor at the University of Budapest, 1957–1964, a Fellow of the Mathematical Institute of the Hungarian Academy of Sciences, 1964–1984, and Professor at the Vienna University of Technology from 1985 until his retirement in1998. He is a member of the Hungarian Academy of Sciences, Academia Europaea, International Statistical Institute, Institute of Mathematical Statistics, and the Bernoulli Society (of which he was President, 1983–1985). His publications include *The Laws of Large Numbers* (Academic Press, New York 1967), *Strong Approximations in Probability and Statistics* (Academic Press, New York 1981, co-authored with M. Csörgõ), *Random Walk in Random and Non-Random Environments* (World Scientific, Singapore 1990), and *Random Walks of Infinitely Many Particles* (World Scientific, Singapore 1994).