ERGODIC PROPERTIES OF STATIONARY, MARKOV, AND REGENERATIVE PROCESSES

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Summary

Ergodic theorems give conditions for the convergence of time averages of stochastic processes. Probably the most natural processes to consider in this respect are stationary processes, and so versions of the ergodic theorem are established for both weak and strict sense stationary processes. Another class of stochastic processes for which ergodic theorems are easily obtained is given by the Markov processes. In addition, regenerative processes, which extend the definition of Markov processes, are defined, and the ergodic theorem is extended to these processes, too. Finally some applications of the ergodic theorem are discussed.

1. Introduction

Ergodic theorems in general are about questions of the convergence of time averages of (not necessarily stochastic) processes. In the case of stochastic processes, there are two parts to a question like this: first, there is the question of existence: there is no need to think any further if the time averages do not converge in the first place. Then, there is the question of the identification of the limit; in the ideal case, the limit would be the mathematical expectation of the associated random variable (if one can give a clear interpretation to this, of course), but things do not always turn out that simple. Anyway, the idea of time averages converging to expectations is one of the cornerstones of statistical mechanics, and, maybe in a somewhat lesser extent, of mathematical statistics.

The archetypical example is the classical law of large numbers: for a sequence $\xi(n)$ of independent, identically distributed random variables with expectation μ, one simply has

$$\frac{1}{n}\sum_{k=1}^{n}\xi(k)\to\mu \tag{1}$$

with probability one as $n\to\infty$. Actually, even the converse is true in the following sense: if the limit above exists, then the variables $\xi(n)$ do have a finite expectation, and it is equal to μ.

Given that one hopes that the time averages will converge to the expectation of the one-dimensional marginal distribution, it is obvious that the most important case is the one where this hope has any chance to be fulfilled, that is, if the one-dimensional marginals are all the same; so, the first case that will be studied is that of a stationary process.

Actually, the statement about the objective of ergodic theory that was made above is, to say the least, incomplete. Modern ergodic theory is far more general, and in particular, it is not restricted to questions from the theory of stochastic processes or from probability theory. Rather it can be described as the theory of the iterates of a positive contraction on some abstract function space, that is, of an operator *T* that satisfies

1. If $f\geq 0,$ then $Tf\geq 0,$ and
2. $\int\|Tf\|\leq\int\|f\|.$

In the case of a stationary process (in discrete time), the role of the operator *T* is, of course taken by the shift operator θ_1 (this operator maps $\xi(t)$ to $\xi(t+1)$ and can be extended to the set of all random variables that can be defined in terms of ξ. For a more formal definition, see *Stationary Processes*. But stationary processes are not the only ones that come along with a natural contraction; the transition operators of a Markov process exhibit the same property.

Thus, Markov processes (more precisely, Markov chains) are another candidate for studies related to ergodic theory. Finally, a slight extension of the notion of a Markov process, the so-called "regenerative processes", which have some importance in applied fields like queuing theory and decision theory, will be introduced, and their ergodic properties will be studied.

2. Ergodic Theory for Stationary Processes

2.1. The Mean Square Ergodic Theorem

As there are two ways in which stationarity can be defined, namely weak stationarity, which only states that the mean and covariance function remain unaffected by a shift time, and strict stationarity, which demands shift invariance for all finite-dimensional marginals, there naturally is more than one way to formulate ergodic theorems for stationary processes. In the case of a weak sense stationary process, the appropriate type of convergence to consider is convergence in mean square. Actually, in *Stationary Processes* a related result has already been proved, which shall be recalled here: Let $\xi(n)$ be a centered weakly stationary sequence, and $Z(.)$ is the associated spectral process (i.e., and orthogonal increment process such that $\xi(n)=\int_{-\pi}^{\pi} e^{int}dZ(t)$), then

$$\lim_{n-m\to\infty}\frac{1}{n-m}\sum_{k=m}^{n-1}\xi(k)=Z(0)-Z(0-). \tag{2}$$

If $\xi(t)$ is a centered weakly stationary process in continuous time, one can likewise prove

$$\lim_{t-s\to\infty}\int_s^t \xi(x)dx = Z(0)-Z(0-). \tag{3}$$

where *Z* is again the associated spectral process.

Thus, the limit (in square mean) of the time average exists and equals the height of the jump of the random function Z in 0. So, a necessary and sufficient condition for the convergence of the mean to the expectation of $\xi(t)$ (which is zero in the case studied here, but the general case is readily settled by considering $\xi(t-m)$ is the continuity of the spectral function *F* at 0.

This criterion is nice, but one would rather have a criterion in terms of the correlation function itself; this goal is not too hard to achieve, and is given by the following theorem:

Theorem 1 *Let $\xi(n)$ be a weak sense stationary sequence with mean m and covariance function $R(.)$; furthermore, let*

$$\bar{\xi}(n)=\frac{1}{n}\sum_{k=0}^{n-1}\xi(k). \tag{4}$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}R(k)=0, \tag{5}$$

if and only if

$$\lim_{n\to\infty}\mathbf{E}((\bar{\xi}(n)-m)^2)=0\,. \tag{6}$$

Without loss of generality, it may be assumed that $m=0$ (otherwise, one can pass on to $\eta(n)=\xi(n)-m$).

It is readily seen that (5) is the limit of the covariance of $\bar{\xi}(n)$ and $\xi(0)$, and that (6) is the limit of the variance of $\bar{\xi}(n)$. Thus, the theorem states that the variance of $\bar{\xi}(n)$ tends to zero if and only if the covariance of $\xi(0)$ and $\bar{\xi}(n)$ tends to zero.

$$\bar{R}(n)=\frac{1}{n}\sum_{k=0}^{n-1}R(k) \tag{7}$$

and observe that by Cauchy's inequality,

$$(\bar{R}(n))^2=(\mathbf{E}(\xi(0)\bar{\xi}(n)))^2\le\mathbf{E}(\xi(0)^2)\mathbf{E}\bar{\xi}(n)^2), \tag{8}$$

so (6) implies (5) .

For the opposite direction, calculate the variance of $\bar{\xi}(n)$:

$$\mathbf{E}(\bar{\xi}(n)^2)=\frac{1}{n^2}\left(\sum_{k=0}^{n-1}\mathbf{E}(\xi(k)^2)+2\sum_{1\le k<l\le n-1}\mathbf{E}(\xi(k)\xi(l))\right)=$$
$$\frac{1}{n^2}\left(\sum_{k=1}^{n}2k\bar{R}(k)-nR(0)\right). \tag{9}$$

By assumption, $\bar{R}(n)\to 0$ as $n\to\infty$, and so, by Kronecker's lemma, the first term above tends to zero as $n\to\infty$. The second term obviously is negligible, so the theorem is completely proved.

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Biographical Sketch

Karl Grill received the Ph.D. degree from TU Wien in 1983. Since 1982 he is with TU Wien where he became an Associate Professor in 1988. He was a visiting Professor in the Department of Statistics, University of Arizona during 1991-92. From February to August 1994, he held NSERC Foreign Researcher Award, Carleton University, Ottawa, Canada