MATHEMATICAL MODELS IN HYDRODYNAMICS

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Contents

Summary

Mathematical modeling plays an important role whenever flow investigations are performed. In this paper special emphasis was given to models which are used in turbulent flow simulations and the effect of numerical discretization. The numerical scheme is of great importance when direct numerical simulations, i.e. the solution of the Navier-Stokes equations without any additional turbulence model, are performed. Applying DNS only flows with low to moderate Reynolds numbers can be studied, because the physics demand for very fine grids in order to resolve even the smallest turbulent scales. Numerical schemes which content artificial dissipation damp the evolution of the turbulent structures and cannot be used in DNS. For the example of the turbulent flow through a straight pipe, the possibilities and the accuracy of DNS were discussed.

On the other hand solving the statistically averaged Navier-Stokes equations (RANS) is the state of the art in applied flow research. In order to be able to solve these RANS equations, turbulence models have to be used. They are based on weak physical assumptions and lead to an incomplete description of the turbulent flow. The results of RANS simulations of turbulent flow in a channel were compared to DNS data. It was further shown that the most popular turbulence models behave physical in non-isotropic wall-bounded flows. Last but not least, the most promising technique for future applied flow research, the large eddy simulation was presented. Different approaches which are used to model the unresolved turbulent fluctuations were discussed. Results of LES with different subgrid scale were compared to measurements. This comparison revealed the reliability of LES for turbulent flows with higher Reynolds numbers.

1. Introduction

Hydrodynamics plays an important role in many environmental and industrial processes. Even in our daily life we experience the transport of water and the associated dynamics in many situations. In order to better understand the physical mechanisms and to take advantage of this knowledge in controlling hydrodynamics a huge number of experimental and theoretical studies were performed in the past. Due to several reasons, which will be discussed below, the end of this search is and probably will not be in sight for some years to come.

During the 20th century many experimental facilities have been built and various measurement techniques were developed to visualize and quantify flows in different configurations. The existing experimental equipment is still intensively used in detailed flow measurements with fundamental and applied objectives. In any measurement technique electrical or optical signals are transformed using mathematical models to obtain the desired flow field. They usually describe simplified physical laws. The reliability of these models often determines the accuracy of the conducted flow measurements. Further, in the 1980s and 1990s research in fluid mechanics focused more and more on the improvement techniques for numerical flow predictions. One expects to cut costs and at the same time increase the available information. Obviously mathematical models play an even more important role in computational fluid dynamics (CFD). The governing equations for most flow problems are usually not solvable by analytical means. In order to obtain a numerical solution they have to be discretized on grids, which intersperse a defined computational domain. A certain discretization method principally already represents a mathematical model, since the associated numerical discretization or truncation errors can in certain cases dramatically alter the generated solution. Some discretization schemes for example hide numerical dissipation, which damps turbulent fluctuations. In this respect these dissipative schemes act like a turbulence model, but they do not include any physical meaning.

Phenomenologically most flows can be categorized into the laminar and the turbulent flow regimes. Any disturbance, which is damped by molecular dissipation in a laminar flow, grows to form a turbulent, chaotic-like, time-dependent, three-dimensional flow field, if the relative impact of the molecular diffusion, which is expressed with the inverse of the Reynolds number $Re = \rho lu/\mu$ (μ and ρ represent the dynamic viscosity and density of the fluid and l and u representing length and velocity scales of the flow) is reduced. For high Reynolds numbers the flow usually develops into a turbulent flow, which is characterized by fluctuations on a wide range of scales. In order to properly resolve all scales in a so-called direct numerical simulation (DNS) one has to specify extremely fine grids. Additionally, the size of the smallest scales decreases with increasing Reynolds number resulting in unaffordable computing times which scale with the third power of the Reynolds number. This makes DNS for high Reynolds numbers flows impossible. To overcome this, researchers developed turbulence models, which approximate the effect of smaller scales. This is necessary since most environmental or technically relevant turbulent flows are characterized by very high Reynolds numbers. Hence, turbulence modeling is one of the key problems in numerical hydrodynamics.

2. Some Fundamentals

The reliability of numerical predictions in hydrodynamics is determined by three components, the mathematical model, the numerical method and the available computer. Unless the Mach number, which is defined as the ratio of a characteristic velocity of the flow to the speed of sound, is exceptionally high, it is generally accepted that the Navier-Stokes equations (1) describe most hydrodynamic flows. They are derived applying the principles of conservation of mass and momentum for a continuum fluid. The continuum assumption holds as long as the size of the smallest flow structures (eddies) is considerably larger than the mean free distance of the fluid molecules.

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \circ \left(\rho\, \vec{u}\right) = 0, \\ &\frac{\partial(\rho\, \vec{u})}{\partial t} + \nabla \circ \left(\rho\, \vec{u}\vec{u}\right) + \nabla \circ p - \nabla \circ \underline{\tau} = 0 \end{aligned} \qquad (1)$$

Eq.(1) represents a system of four time-dependent non-linear partial differential equations for the velocity vector $\vec{u}$, the pressure p and density ρ. In Cartesian coordinates (x, y, z) the corresponding velocity vector is $\vec{u} = (u, v, w)$ and the nabla operator $\nabla = ({\partial}/{\partial x}, {\partial}/{\partial y}, {\partial}/{\partial z})$. To obtain full closure for Eq. (1) additional equations like the energy equation and the equation of state have to be solved.

For simplicity we consider a Newtonian, incompressible fluid with constant density ρ and constant dynamic viscosity μ for which the viscous stress tensor $\underline{\tau}$ is directly proportional to the strain rate tensor as expressed in Eq. (2).

$$\underline{\tau} = \mu(\nabla\vec{u} + (\nabla\vec{u})^{\mathrm{T}}) \qquad (2)$$

If we additionally introduce representative velocity and length scales u_{ref} and l_{ref} to non-dimensionalize Eq. (1) and Eq. (2), one obtains the dimensionless incompressible Navier-Stokes equations (3), which contain the dimensionless Reynolds number *Re*.

$$\begin{aligned} &\nabla \circ \vec{u} = 0, \\ &\frac{\partial \vec{u}}{\partial t} + \nabla \circ \left(\vec{u}\vec{u}\right) + \nabla \circ p - \underbrace{\frac{\nu}{u_{\mathrm{ref}}\, l_{\mathrm{ref}}}}_{1/Re} \nabla^2 \vec{u} = 0 \end{aligned} \qquad (3)$$

Depending on the value of the Reynolds number Re in Eq. (3) solutions are obtained either in the laminar or in the turbulent flow regime. Laminar flows are characterized by their smooth and regular nature. Most known analytical solutions of Eq. (3) are valid only in the laminar regime, i.e. for low Reynolds numbers. For higher Reynolds numbers disturbances, which are introduced for example by imperfect or rough walls, tend to grow into three-dimensional vortical structures. Due to stretching and tilting of the associated vortex lines an initially simple flow pattern changes into complicated turbulence, which is characterized by irregularity in space and time, increased dissipation, mixing and non-linearity.

Often, one is not interested in all details of the behavior of the time-dependent three dimensional velocity and pressure field, but in their mean behavior. To obtain the mean flow field any turbulent variable f is split into a statistical mean $< f >$ and its fluctuation $f^{"}$ as expressed in Eq. (4).

$$f = < f > + f^{"} \quad \text{with}$$

$$< f > = \lim_{T \to \infty} \int_{t_0}^{t_0+T} f dt \tag{4}$$

Often only simple statistics like the mean value and rms-value $f_{\mathrm{rms}} = \sqrt{f^{"2}}$ are required for engineering or geophysical purposes. In practice the statistical mean values are either calculated by averaging with respect to time t (see Eq. (4)), or by averaging in one or more spatial directions, if the turbulent flow field can be considered to be homogeneous in those directions. Typical examples are the fully developed turbulent flow in a straight pipe or the fully developed turbulent channel flow, which is the flow between two horizontally extended plates driven by a horizontal pressure gradient. For both flows periodic boundary conditions in axial and azimuthal or spanwise directions can be applied, since the flow can be considered to be homogeneous in these two directions.

Statistically averaging Eq. (3) leads to the so-called Reynolds-averaged Navier-Stokes (RANS) equations. In Eq. (5) they are presented in Cartesian coordinates using Einstein's summation convention.

$$\begin{aligned}
&\frac{\partial < u_i >}{\partial x_i} = 0, \\
&\frac{\partial < u_i >}{\partial t} + \frac{\partial < u_i >< u_j >}{\partial x_j} + \frac{\partial < u_i^{"} u_j^{"} >}{\partial x_j} \\
&= -\frac{\partial < p >}{\partial x_i} + \frac{1}{Re} \frac{\partial}{\partial x_j} (\underbrace{\frac{\partial < u_i >}{\partial x_j} + \frac{\partial < u_j >}{\partial x_i}}_{2S_{ij}})
\end{aligned} \tag{5}$$

Compared to Eq. (3), Eq. (5) contains an extra non-linear term, the so-called Reynolds stress tensor $< u_i^{"} u_j^{"} >$.

The turbulent kinetic energy k, which is defined by the trace of the Reynolds stress tensor $k = \frac{1}{2} < u_i^{"} u_i^{"} >$, is produced from the mean flow, which does the stretching (if a mean strain rate is present). This energy is successively passed down from large scale vortex motions down to motions of smaller and smaller scales until it is finally dissipated into thermal energy. This process is known as the energy cascade. It serves as a basis for many mathematical models which are used in turbulent flow predictions. Considering isotropic turbulence for example with $< u_1^{"2} >=< u_2^{"2} >=< u_3^{"2} >$ and assuming a universal equilibrium, Kolmogorov derived the one-dimensional energy spectrum Eq. (6)

$$E(k) \sim \varepsilon^{2/3} k^{-5/3} \qquad (6)$$

which states that the energy of the vortical structures (eddies) in isotropic turbulence is proportional to the $-5/3$ power of their wave-number k if viscous effects, expressed by the dissipation rate ε, are significant. The typical length scale of the dissipating, i.e. the smallest, eddies is the so-called Kolmogorov scale $K = (\nu^3/\varepsilon)^{1/4}$. A simplified sketch of a Kolmogorov spectrum in a double logarithmic plot over the wave number k is presented in Fig. 1.

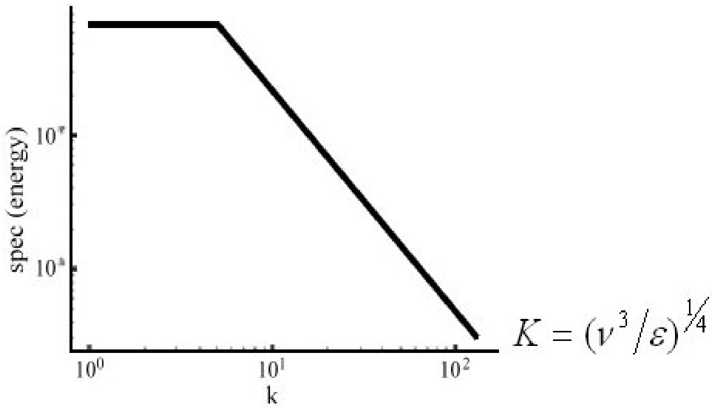

Figure 1: Simplified sketch of a Kolmogorov spectrum

Principally, there are three different approaches to predict turbulent flows numerically. Due to the dramatically increased efficiency of the super computers and new numerical methods it is possible to solve the time-dependent Navier-Stokes equations without any turbulence model in a Direct Numerical Simulation (DNS). However, the huge computational resources, which are required to conduct a DNS restricts their application to turbulent flows at lower Reynolds numbers with in most cases just an academic objective. The obtained turbulent flows though are characterized by random three-dimensional fluctuations with a continuous spectrum of length scales ranging down to flow structures which dissipate the excessive energy, the Kolmogorov scales. This will be demonstrated in Section 3 presenting results produced in DNS of the fully developed flow in a straight pipe.

Turbulent flow calculation with a more applied objective were and are still performed solving the Reynolds averaged Navier-Stokes equations (Eq. (5)). In many cases the obtained solution is stationary and, depending on the number of homogeneous directions involved, one or two dimensional. The statistical approach is associated with the highest loss of information and with a closure problem which is not satisfactorily solved. Spectral information is completely lost, since any statistical quantity is an average over all turbulent scales. The obtained flow field describes the mean flow, which is enough for many applied problems, while the turbulent information is described with the Reynolds stress tensor. This tensor has to be modeled with empirical or semi-empirical models. There are a huge number of different turbulence models, because so far there is no generally valid statistical turbulence model. In Section 4 the popular (k,ω)-turbulence model will be discussed based on results which were obtained in simulations of the fully developed turbulent channel flow.

In the 1990s the interest in time-dependent, three-dimensional analysis of turbulent flows increased, since many technical problems are associated with large scale motions. To obtain these time-dependent flow structures researchers more and more conducted unsteady RANS simulation on three dimensional meshes with increasing number of grid points. Doing this they pushed the RANS technique more and more towards the third approach in numerical turbulence simulation, the Large-Eddy Simulation (LES). This method basically resembles a compromise between RANS and DNS since it allows predicting the dynamics of the large turbulent scales while the effect of the fine scales are modeled with a subgrid-scale model. The most common subgrid scale models are discussed in Section 5 presenting LES results of the turbulent pipe flow.

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Biographical Sketch

Claus Wagner, is Head of the Section “Numerical Simulation of Technical Flows” at the German Aerospace Center (DLR) in Göttingen, Germany and Professor of Industrial Aerodynamics at the University of Technology in Ilmenau, Germany. Professor Wagner’s research and teaching interests are focused in a number of areas. One is thermal convection, where Wagner concentrates on the numerical simulations of turbulent free and mixed convection in containers and aircraft cabins and their experimental validation. A second area is the development and application of sophisticated numerical methods for Direct Numerical Simulation and Large Eddy Simulations of turbulent flows. A third focus is the development and evaluation of turbulence models for Large-Eddy Simulations. Of special interest is the application of the advanced numerical techniques for the prediction of flow with an applied objective in close cooperation with aircraft manufacturers, as well as with the railroad and semiconductor industry. Professor Wagner grew up in South Germany, studied in Munich, Germany and lived in the USA for more than 2 years conducting research at the University of Florida in Gainesville, Florida, USA and at the NASA Langley Research Center in Hampton, Virginia, USA. He is now living in Göttingen with his family.