DISCRETE OPTIMIZATION

Franz Rendl
Institut für Mathematik, Universität Klagenfurt, Austria

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Contents

Summary

Optimizing a linear function over a discrete set is the topic of Discrete Optimization. Standard modeling techniques leading to discrete problems are presented, along with solution methods. These are based on relaxation and the design and analysis of heuristics. Linear Programming plays a fundamental role as relaxation. The Greedy algorithm, together with randomized improvement methods, are the key tools for heuristics.

1. Introduction

A policy maker selecting among many alternatives may use economical criteria to come to the 'best' decision. Often this process can be modeled by yes-no decisions. Discrete optimization is the discipline that connects operations research modeling and mathematical optimization methods. In discrete optimization, one tries to devise operations research models that are computationally tractable. Thus, the modeling phase is biased towards models for which practically acceptable algorithms exist. On the other hand, theoretical research focuses on broadening the mathematical machinery to deal with more complicated models.

This survey reviews the most relevant modeling techniques in Section 2, and the most reliable mathematical solution methods in Section 3.

2. Modeling

Budget planning in its simplest form can be modeled as follows. Suppose we have a budget B available, which we may use for investment in activities 1, …, n. The cost of activity i is known to be c_i, its profit p_i. If we denote the set of all activities by $N := \{1, \ldots, n\}$, we can find an optimal investment strategy by solving the following problem.

$$\text{maximize} \sum_{i \in N} p_i x_i \quad \text{such that} \sum_{i \in N} c_i x_i \le B,\ x_i \in \{0,1\}\ \forall\ i. \tag{1}$$

The decision variable x_i takes the value 1 if we invest in activity i. The inequality constraint expresses the fact that we cannot invest beyond our budget B.

This type of problem is called an *integer linear program*, because we optimize a linear function, subject to linear constraints, over integer variables. In the optimization literature, an integer linear program with only one constraint is called the *Knapsack problem*.

Many problems in management science and operations research, where the objective is to make the most economical decision among a set of given alternatives, can be formulated as integer linear programs. We will now describe some of the standard model prototypes. A further interesting feature of integer linear programs is that they also allow one to model logical conditions. Finally, going beyond linear functions can lead to interesting and still manageable prototype models.

2.1 Linear Models

The *Assignment Problem* is among the simplest, and most studied integer linear programs. It can be stated as follows. We are given a set of n jobs $J_1, \ldots, J_n$ and m workers $W_1, \ldots, W_m$. A profit of p_{ij} occurs, if worker W_i carries out job J_j. If we wish to assign (some of) the jobs to workers, so that each job is done at most once, and each worker handles at most one job, and at the same time (thus, maximizing total profit); we end up with the assignment problem:

$$\text{maximize} \sum_{ij} p_{ij} x_{ij} \ \text{such that} \sum_{i} x_{ij} \le 1\ \forall\ j \ \text{and} \sum_{j} x_{ij} \le 1\ \forall\ i,\ x_{ij} \in \{0,1\}. \tag{2}$$

The decision variables x_{ij} express whether worker W_i carries out job J_j (x_{ij} = 1) or not (x_{ij} = 0).

Sometimes it is more important to consider the amount of time that is spent until the last job is finished, assuming all jobs are started at the same time. This variant of the assignment problem can be modeled as follows. Instead of looking at the profit, we consider t_{ij}, the time needed by worker W_i to handle job J_j. If we wish to assign all jobs so that they are completed as quickly as possible, we arrive at the *Bottleneck Assignment Problem*:

$$\text{minimize} \max_{ij}\ t_{ij} x_{ij} \ \text{such that} \sum_{i} x_{ij} = 1\ \forall\ j \ \text{and} \sum_{i} x_{ij} \le 1\ \forall\ i,\ x_{ij} \in \{0,1\}. \tag{3}$$

This problem is only meaningful if $m \ge n$, otherwise there are not enough workers to execute the jobs. We also observe that the objective function is only piecewise linear, but the resulting problem can still be solved efficiently. Another useful prototype model is the *Set-Covering Problem*. Suppose there are k villages $V = \{v_1, \ldots, v_k\}$ that wish to set up fire stations to cover all the villages. To minimize costs, it is decided that a fire station located in village v_i also can serve (aside from v_i itself) nearby villages, which are reachable within some short time limit. We denote this set of villages by S_i. Let us introduce the matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & v_i \in S_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The cost of building a fire station in village v_i is known to be c_i. The problem,

$$\text{minimize} \sum_i c_i \text{ such that } \sum_j a_{ij} x_j \geq 1 \;\forall\; i, x_i \in \{0,1\}. \tag{5}$$

yields the most economical selection of fire stations, while at the same time ensuring that each village is reachable from some nearby station.

Finally, we briefly describe the *Cutting Stock Problem*. A plumber has an order for plumbing a building. He uses pipes of fixed diameter but needs pieces of various lengths. Specifically, he needs b_i pieces of length l_i, ($i = 1, \ldots, k$). As raw material, he has pipes of length L, where $L \geq l_i \forall i$. He is faced with the question of how to cut the raw pieces of length L, which are available in a sufficient number, to the required lengths, in order to minimize total waste. The mathematical model for this type of problem is less straightforward than for the previous prototypes. A cutting pattern indicates how many pieces of type i are cut from one raw piece of length L. Suppose that in the cutting pattern j, we cut a_{ij} pieces of type i. For the cutting pattern j to be feasible, it is clearly necessary that

$$\sum_i a_{ij} l_i \leq L. \tag{6}$$

After having generated a set of patterns, we introduce an integer variable x_j for each pattern, which indicates how often pattern j is used. To satisfy the demand, it is necessary that

$$\sum_j a_{ij} x_j \geq b_i \;\forall\; i = 1, \ldots, k. \tag{7}$$

Minimizing waste amounts to

$$\text{minimize} \sum_i x_i \text{ subject to } \sum_j a_{ij} x_j \geq b_i \;\forall\; i = 1, \ldots, k, x_j \geq 0 \text{ and integer.} \tag{8}$$

This is again an integer linear program, but this time, the decision variables x_j are not limited to 0 or 1.

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Biographical Sketch

Franz Rendl studied Technical Mathematics in Graz, where he received his diploma in 1980. He joined the Mathematics Department of the Technical University Graz in 1982, where he worked in the group of Discrete Optimization. Since 1998, he is full professor at the University of Klagenfurt, heading a research group in Operations Research. His scientific interests focus on the interplay between discrete optimization problems and structured convex optimization. He has experience with practical implementations of Operations Research models in industry. He is in the editorial board of several major journals in optimization and Operations Research, such as SIAM Journal on Optimization, Mathematical Programming and Operations Research Letters.