# COMBINATORIAL OPTIMIZATION AND INTEGER PROGRAMMING

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## Contents





## Summary

Solution techniques for combinatorial optimization and integer programming problems are core disciplines in operations research with contributions of mathematicians as well as computer scientists and economists. This article surveys the state of the art in solving such problems to optimality.

## 1. Introduction

Combinatorial optimization and integer programming is concerned with finding optimum solutions for optimization problems that involve yes/no decisions or determining optimum levels of discrete quantities. Research in solution techniques and corresponding computer software originated in the fifties and has been flourishing especially in the last decade. Our overview of the current state of the art is organized as follows.

In Section 2, we introduce a few illuminating example problems. We also generalize the generic models that are the subject of this article. In Section 3, we sketch the mathematical foundations of today's solution techniques. Section 4 deals with the most important algorithmic approaches. Section 5 concludes our exposition with general remarks on the availability of these techniques as computer software.

## 2. Modeling

### 2.1. Example Applications

Many of the most well-known operations research problems can be formulated as (mixed) integer linear programs. Before both generic (mixed) integer programming models and generic combinatorial optimization models are introduced, a few examples are considered.

In the *assignment problem* (*AP*), persons must be assigned to jobs, say, $n$ persons to $n$ jobs. If $p_{ij}$ denotes the level of proficiency person $i$ possesses for job $j$, the task is to find an assignment that maximizes the total proficiency. Let the unknown $x_{ij}$ be either 1 or 0 if person $i$ is assigned to job $j$ or not, respectively. Then the problem can be written as

$$
\begin{aligned}
&\max \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}x_{ij} \\
&\qquad \sum_{i=1}^{n} x_{ij} = 1, \qquad \text{for all } j \in \{1, 2, ..., n\}, \\
&\qquad \sum_{j=1}^{n} x_{ij} = 1, \qquad \text{for all } i \in \{1, 2, ..., n\}, \\
&\qquad x_{ij} \in \{0,1\}, \qquad \text{for all } i, j \in \{1, 2, ..., n\}.
\end{aligned}
\tag{1}
$$

The equations make sure that each person is assigned to exactly one job and each job to exactly one person.

For a more compact formulation of the assignment problem (and some of the subsequent example problems) it is convenient to introduce *undirected graphs* $G = (V, E)$ with a finite *node* set $V$ and *edge* set $E \subseteq \{\{u, w\} \mid u, w \in V, u \neq w\}$. Two edges are *adjacent* if they share a common node. The complete bipartite graph $K_{n,n}$ is the graph $K_{n,n} = (V, E)$ with node set $V = U \cup W$, where $|U| = n$, $|W| = n$, $U \cap W = \emptyset$, and edge set $E = \{\{u, w\} \mid u \in U, w \in W\}$. Let $\mathbb{R}^E$ denote the real vector space of dimension $|E|$ where the components are indexed by the elements of $E$. For a set $F \subseteq E$ its *characteristic vector* $x^F \in \mathbb{R}^E$ is defined by setting $x_e^F = 1$, if $e \in F$, and $x_e^F = 0$, otherwise. An assignment then is a set of $n$ pairwise nonadjacent edges of $K_{n,n}$ and the $x_{ij}$ become its characteristic vector.

For a graph $G = (V, E)$ and a node set $W \subseteq V$ let $\delta(W) := \{\{u, w\} \in E \mid |\{u, w\} \cap W| = 1\}$ denote the *cut* induced by $W$, i.e. the edges with one endnode in $W$ and the other in $V \setminus W$, and let $\delta(v)$ be a shorthand for $\delta(\{v\})$. For an edge set $F \subseteq E$ and variables $x_e$ ($e \in E$), let $x(F) := \sum_{e \in F} x_e$. Furthermore, for $a, b \in \mathbb{R}^n$, let $ab := \sum_{i=1}^n a_i b_i$ denote the inner product of $a$ and $b$. Then the assignment problem can equivalently be written as

$$\begin{aligned} &\max\ px \\ &x(\delta(v)) = 1, \quad \text{for all } v \in V, \\ &x_e \in \{0,1\}, \quad \text{for all } e \in E. \end{aligned} \tag{2}$$

The related *perfect matching problem* (*PMP*) arises if one wants to find an optimum pairing of an even number $2n$ of items where each pairing of $i$ and $j$ induces a profit $p_{ij}$. E.g., assigning students to double rooms in a dormitory is such a task where there is no bipartition like in the assignment problem. In graph theoretic terms, the problem is to determine $n$ pairwise nonadjacent edges in a complete graph $K_{2n} = (V, E)$ where $|V| = 2n$ and $E = \{\{u, w\} \mid u, w \in V, u \neq w\}$, with edge weights $p_{ij}$. It can be modeled just like (1) with an underlying complete graph $K_{2n}$ instead of a complete bipartite graph $K_{n,n}$.

A traveling salesman, starting in his home city, must visit each of additional $n - 1$ cities exactly once and return to his home. If $d_{ij}$ denotes the distance between towns $i$ and $j$ (where $d_{ij} = d_{ji}$), one of his problems, the *traveling salesman problem* (*TSP*), consists of choosing a tour of minimum distance traveled.

The problem can be modeled on a complete graph with edges corresponding to the direct connections between two cities weighted according to the distances. Variables $x_{ij} \in \{0, 1\}$ are introduced with the interpretation that $x_{ij} = 1$ if the salesman uses the edge between $i$ and $j$ and $x_{ij} = 0$ otherwise. Using the notation $E(W) = \{\{i, j\} \mid i \in W, j \in W, i \neq j\}$, the task can then be formulated as

$$\begin{aligned} &\min\ dx \\ &x(\delta(v)) = 2, \quad \text{for all } v \in V, \\ &x(E(W)) \le |W| - 1 \quad \text{for all } W \subset V,\ 3 \le |W| \le n-2, \\ &x_e \in \{0,1\}, \quad \text{for all } e \in E. \end{aligned} \tag{3}$$

The equations, called the *degree constraints*, express that in a tour, each city is touched by two direct connections; i.e. each node of the graph is incident with exactly two tour edges. The inequalities, called *connectivity constraints*, make sure that each nonempty subset of cities other than the whole set $V$ is entered and left, thus excluding short cycles.

Let $N = \{1, 2, \dots, n\}$ be a set of potential fire station locations and $M = \{1, 2, \dots, m\}$ a set of communities to be protected. For $j \in N$ let $M_j \subseteq M$ denote the set of communities that can be reached from location $j$ in less than 10 minutes. Then, let $c_j$ denote the cost of building a fire station at location $j$. The task, to decide which stations to build at the least possible cost so that all communities are protected, can be formulated as a *set covering problem* (*SCP*).

Let $B = (b_{ij}) \in \mathbb{R}^{m\times n}$ be the matrix whose columns are the characteristic vectors of the sets $M_j$, $j = 1, 2, \ldots, n$, i.e., $b_{ij} \in \{0, 1\}$ and $b_{ij} = 1$ if and only if community $i$ can be protected by station $j$. If **1** denotes the vector of all 1’ s, then one has to solve the following problem.

$$\begin{aligned} &\min\ cx \\ &Bx \geq \mathbf{1} \\ &x \in \{0,1\}^n. \end{aligned} \tag{4}$$

This type of problem belongs to the broad class of location problems.

For the classical *facility location problem* (*FLP*), the input consists again of a set $N = \{1, 2, \ldots, n\}$ of potential facilities and a set $M = \{1, 2, \ldots, m\}$ of clients with demands of certain goods supplied from the facilities.

Facility $j \in N$, if built at cost $c_j$, has a capacity of $u_j$. The demand of client $i \in M$ is $b_i$. It costs $h_{ij}$ to satisfy a unit of $i$’s demand from facility $j$. If $x_j \in \{0, 1\}$ encodes the decision of whether or not facility $j$ is opened, and the continuous variable $y_{ij}$ the quantity of $i$’s demand that it is satisfied from facility $j$, then the problem is the following.

$$\begin{aligned} \min \sum_{j\in n} c_j x_j + \sum_{i\in M}\sum_{j\in N} h_{ij} y_{ij} & \\ \sum_{j\in N} y_{ij} = b_i, \quad & \text{for all } i \in M, \\ \sum_{i\in M} y_{ij} - u_j x_j \leq 0, \quad & \text{for all } j \in N, \\ y_{ij} \geq 0, \quad & \text{for all } i \in M,\ j \in N \\ x_j \in \{0,1\}, \quad & \text{for all } j \in N. \end{aligned} \tag{5}$$

The equations guarantee that all demands are satisfied and the inequalities ensure that only opened facilities are used and their capacities are not exceeded.

A picture frame consists of two horizontal and two vertical parts. A picture frame manufacturer needs to cut $m$ frame parts of lengths $a_1, a_2, \ldots, a_m$ from base rods of length $L$. For producing the parts, he wishes to determine a cutting strategy that minimizes the number of used base rods.

Defining $M$ and $N$ as above, one can use analogous techniques as in (1)-(5) to formulate the problem. Let $n \leq m$ be an upper bound on the number of required base rods. Then, the so-called *cutting stock problem* (*CSP*) is the following.

$$
\begin{aligned}
& \min \sum_{j \in n} x_j \\
& \sum_{j \in N} y_{ij} = 1, && \text{for all } i \in M, \\
& \sum_{i \in M} a_i y_{ij} - L x_j \leq 0, && \text{for all } j \in N, \\
& y_{ij} \in \{0,1\}, && \text{for all } i \in M,\ j \in N \\
& x_j \in \{0,1\}, && \text{for all } j \in N.
\end{aligned}
\tag{6}
$$

In any feasible solution $x_j = 1$ if and only if base rod $j$ is used and $y_{ij} = 1$ if and only if part $i$ is cut from base rod $j$.

Two further basic problems are of interest in combinatorial optimization. Like the examples above, they can be formulated in terms of maximizing or minimizing a linear objective function subject to linear constraints and integrality conditions, yet such a description is of less interest here.

A $(v_1, v_k)$–*path* in $G = (V, E)$ is a set of edges $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\}$ with distinct nodes $v_1, v_2, \ldots, v_k \in V$. A graph is *connected* if either $|V| = 1$ or for each pair of distinct nodes $u, w \in V$ there exists a $(u, w)$-path in $G$. A *cycle* arises if all nodes are distinct except $v_1 = v_k$.

Both problems are formulated for an undirected graph $G$ with edge weights. In the *minimum spanning tree problem* (*MSTP*), the objective is to find a *spanning tree*, i.e., a connected subgraph with no cycles (i.e., with $n - 1$ edges) of minimum total weight. In the $(u, w)$-*shortest path problem* (*ShPP*), two vertices $u, w \in V$ are given and the objective is to determine a $(u, w)$-path in $G$ of minimum total weight.

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**Biographical Sketches**

**Michael Jünger** studied computer science and operations research at the University of Bonn and Stanford University (M.S. 1980), and applied mathematics at the University of Augsburg (Dr. rer. nat. 1983). From 1983 to 1990, he was a Research Assistant at the University of Augsburg. From 1990 to 1991, he was an Associate Professor of Mathematics at the University of Paderborn, and since 1991 he has been Professor of Computer Science in the Computer Science Department of the University of Cologne. Dr. Jünger is Co-Editor of *Mathematical Programming*, and Associate Editor of *Mathematical Methods of Operations Research*. His major field of interest is combinatorial optimization, in particular, the design, analysis, implementation and evaluation of algorithms for hard combinatorial optimization problems.

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