HISTORY OF TRIGONOMETRY TO 1550

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Keywords: Trigonometry, astronomy, sine, tangent, spherical astronomy, interpolation, astronomical timekeeping, *qibla*, practical geometry

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Summary

Trigonometry was born from the ancient practice of mathematical astronomy, especially the modeling of the motions of celestial objects. In order to be able to use their geometric models of the planets to predict their positions, Hipparchus and later Ptolemy needed mathematical tools that convert arc lengths to distance measurements. Much of this work took place on the sphere rather than the plane, with Menelaus's Theorem as the fundamental tool. In India trigonometry took on a more computational and less geometric form than its Greek inheritance, but was used for similar purposes. Interpolation and approximation allowed them to solve trigonometric problems that were otherwise unsolvable. Astronomers in medieval Islam, originally influenced by India, embraced Ptolemy's approach when his works were eventually translated into Arabic. However, starting in the late 10th century they began to reformulate the foundations especially of spherical trigonometry, to create a more streamlined and easily applied theory. Trigonometry was used in Islam for astronomical timekeeping, as well as ritual purposes such as determining the direction of Mecca and the beginning of the lunar month. The West was originally dominated by influences passing through Muslim Spain. The "science of triangles" became mathematically rigorous (yet still a servant of astronomy) with Regiomontanus's 15th-century *De triangulis omnimodis*. Finally, the challenge of calculating accurate values of trigonometric functions came to a climax with the massive six-function tables by Rheticus, Otho and Pitiscus in the late 16th century.

1. Precursors

Trigonometry has existed in various forms for at least two millennia and through several distinct cultures. Since it has changed markedly over the centuries, it is a challenging problem to define precisely what it is. As we shall see, trigonometry first emerged as an astronomical tool when the geometry of planetary models came together with the

Babylonian base 60 positional number system. This combination allowed astronomers to convert systematically between arbitrary arcs or angles on circles and corresponding chord lengths within them.

It is possible to identify various earlier events that qualify either as prehistory, or as prerequisites for the development of trigonometry proper. The earliest of these is five problems involving slopes and distances on pyramids, found in the Egyptian Rhind papyrus (1650 BC). The *seqed* of a pyramid is the horizontal displacement of its sloped surface, measured in palms, for every seven palms of vertical displacement (Figure 1). Problems 56-60 of the Rhind papyrus demonstrate how to compute the *seqed* given various pieces of information about the pyramid. It seems that the *seqed* may also have been operating in various contexts in Egyptian art and architecture. However, although the dimensions of triangles are handled in these problems, there is no notion of angles, arcs, or circles that would allow us to identify *seqed* calculations with what came later; nor did the concept find its way elsewhere.

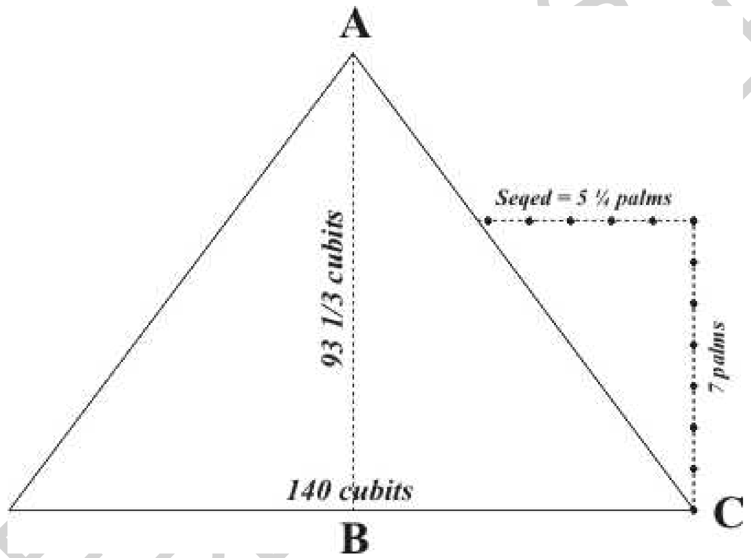

Figure 1. Rhind Mathematical papyrus Problem 58, illustrating the *seqed*

In Babylon we find a similar situation with the famous tablet Plimpton 322 (1700 BC). This controversial document contains a list of *Pythagorean triples*: trios of whole numbers that satisfy the Pythagorean relation $a^2 + b^2 = c^2$. The triples are ordered according to an increasing ratio of a/b, which has led some to suggest that the tablet was used to measure right-angled triangles. But there is no explicit statement of that sort in the tablet, there is no concept of arc or angle at this time in Babylonian history, and there are other more plausible interpretations of the tablet.

However, there are aspects of Babylonian astronomy that set the stage for future developments. The sexagesimal number system, which arose from their system of weights and measures, afforded to astrologers/astronomers the power of positional notation beginning around the 8th century BC. This led to an innovation that we still use today. The ***ecliptic***, the arc in the sky that carries the sun in its annual orbit around the earth, was divided into 12 zodiacal signs, each corresponding roughly to a solar journey of one month. Each sign was further divided into 30 parts, corresponding roughly to a day. Thus the ecliptic was divided into 360 parts, later to be named degrees.

Equipped thus, Babylonian astronomers were able to perform great computational feats that led to successful modeling of the motions of celestial objects — although without the benefit of physical or geometric models. These schemes were strictly arithmetical, not geometrical, and thus do not apply any trigonometry.

On the other hand, early Greek astronomers used geometric models, but had no real quantitative apparatus to accompany them. Works like Autolycus's *On a Moving Sphere*, Euclid's *Phenomena*, and later Theodosius's *Spherics* studied the geometry of the sphere to gain insight into astronomical phenomena, such as *rising times* (the time it takes an arc of the ecliptic to rise above the horizon) and the relations between different coordinate systems on the celestial sphere. But results in the early science of spherics were qualitative rather than quantitative: for instance, certain arcs of the ecliptic have longer rising times than others.

Quantitative mathematical astronomy in ancient Greece did not begin overnight; its origins may be identified with several possible candidates. Two of the earliest, Aristarchus of Samos (ca. 310 BC – 230 BC) and Archimedes of Syracuse (287 BC – 212 BC), worked without the convenience of a positional number system. Aristarchus is known particularly for his heliocentric system, but the work that survives, *On the Sizes and Distances of the Sun and Moon*, deals with an unrelated topic. In this treatise Aristarchus attempts to find the relative distances of the sun and moon from the earth, using an observation of the angular displacement of the two celestial bodies when the Moon is at half phase (Figure 2).

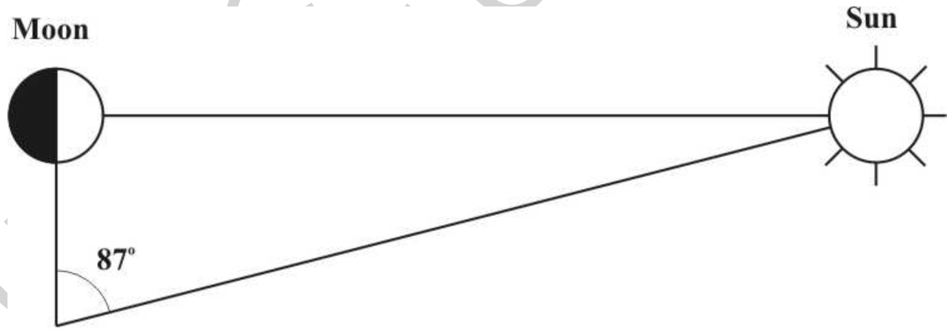

Figure 2. Earth, sun and moon at half moon, from Aristarchus's *On the Sizes*

To convert the angular datum in his observation into a relation between distances, Aristarchus calls upon a lemma that posits an inequality between a ratio of angles on the one hand, and a ratio of lengths on the other. In the two triangles in Figure 3, $AC = DF$ and $\alpha > \beta$; then

$$\frac{EF}{BC} < \frac{\alpha}{\beta} < \frac{DE}{AB}. \qquad (1)$$

This result did not allow Aristarchus to convert angles into lengths with precision, but it did at least give him upper and lower bounds for the lengths he needed.

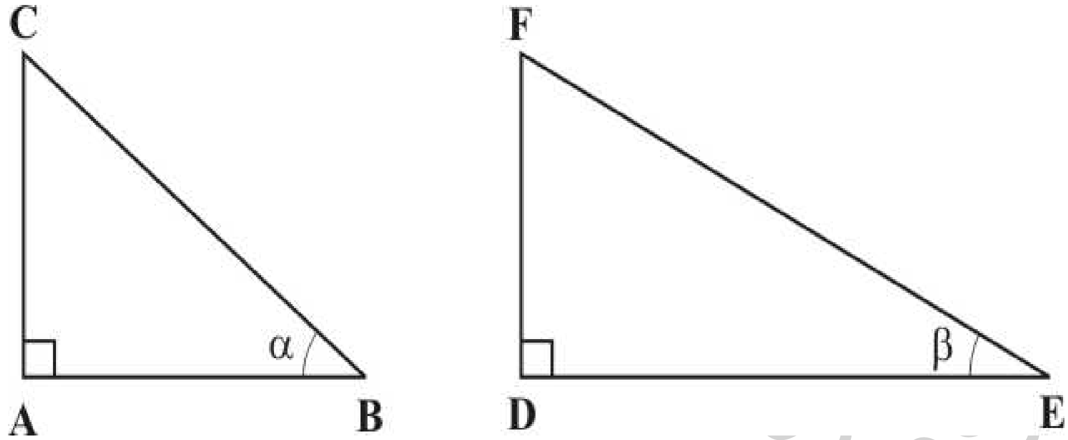

Figure 3. An ancient trigonometric lemma

It is in Archimedes' *The Sand Reckoner* that we read of Aristarchus's heliocentric theory: since the goal of this work is to determine the largest possible number of grains of sand that it would take to fill the universe, Archimedes needs to know the largest estimate for the size of the universe.

A heliocentric theory requires stellar parallax (if the earth moves around the sun, the stars should appear to wobble with a period of one year), and since none is observed, such a universe must be large enough for stellar parallax to be negligible.

In *The Sand Reckoner* Archimedes uses the same lemma that Aristarchus had used, this time with the intent of converting from the sun's angular diameter to upper and lower bounds for its diameter in distance.

The extent to which Archimedes was able to practice trigonometry itself is a matter for interpretation and debate. His uses of the lemma are ingenious, but they are qualitatively different from the systematic abilities to manipulate arcs and lengths that we find later.

However, one Archimedean proposition discovered in the work of the 11th-century Muslim scientist al-Biruni (al-Bīrūnī) is suggestive. The Theorem of the Broken Chord asserts the following (Figure 4): given unequal arcs $\widehat{AB} > \widehat{BC}$ in a circle, let *D* be the midpoint of $\widehat{AC}$ and drop perpendicular *DE* onto *AB*. Then $AE = EB + BC$.

From this theorem one can prove a number of trigonometric identities, including chord equivalents of the sine sum and difference formulas. However, we have no evidence that trigonometry was Archimedes' intent for this result; we may only assert that if he did turn his efforts in a trigonometric direction, he would have had the mathematics at his disposal to pull it off.

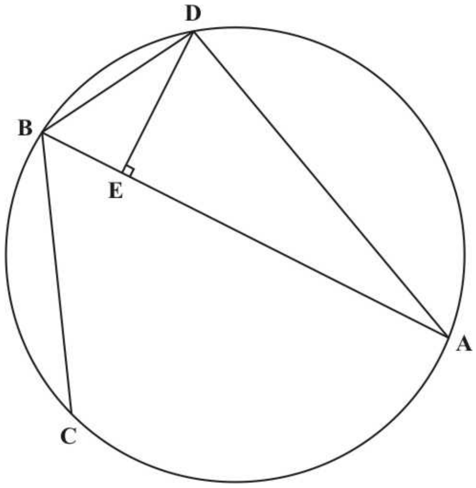

Figure 4. Archimedes's Theorem of the Broken Chord

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Biographical Sketch

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