BOURBAKI, AN EPIPHENOMENON IN THE HISTORY OF MATHEMATICS

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Summary

After the emergence of axiomatizations in mathematics, continuing David Hilbert's breakthroughs performed in geometry for instance, during the 1930s a group of mathematicians started building a coherent axiomatized presentation of major parts of mathematics. The polycephalic monumental Bourbaki volumes have been very influential during half a century at least. Their mode of presentation was widely adopted, but did also provoke controversial situations. The present article provides a large view of the mathematical knowledge displayed all along the Bourbaki treatise.

1. Introduction

The constitution of Bourbaki was one of the most striking and the most influential moments in the history of the XXth century mathematics. The phenomenon is quite unique due to its conception as well as its exceptionally long duration.

2. The Origins

After Augustin - Louis Cauchy's classical milestone *Cours d'Analye(de l'École polytechnique)* several great treatises on Analysis were published in French: Jean Bertrand *Traité de calcul différentiel et intégral*, Camille Jordan *Cours d'analyse de l'École polytechnique*, Émile Picard *Traité d'analyse*, Édouard Goursat *Cours d'analyse mathématique*, Charles de La Vallée Poussin *Cours d'analyse infinitésimale*.

But in the late XIXth century central mathematical innovating subjects began focusing on abstract themes. Georg Cantor gave rise to Set Theory. David Hilbert was the main defender of axiomatization and formalization; his book on *Grundlagen der Geometrie* (Fundamentals of Geometry) became a standard reference.

During World War One fewer German mathematicians were killed on the battle field than their French and even British counterparts. In Paris, after these tragedies, the influential Jacques Hadamard and Bertand Julia seminars became among the rare places where up-to-date mathematics was presented and discussed.

In 1934 a group of young former students at the École Normale Supérieure in Paris started an adventure in view of producing a modern presentation of mathematics that should become a convenient textbook but turned out to be a synthesis, not an encyclopedia, covering a lot of mathematics in an abstract language.

As this treatise was due to cover the main mathematical subjects the authors chose the title *Éléments de mathématique* in order to give credit to Euclid for a previous attempt and to claim that mathematics is a whole, makes up a unit, so going back to the period before d'Alembert when in French one was speaking of *la Mathématique* and not of *les Mathématiques*.

Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonnné and André Weil became the principle persons of the group; but during the first stage important contributions were made also by René de Possel, Charles Ehresmann, Szolem Mandelbrojt, and the physicist Jean Coulomb.

A pseudo name had to be chosen for the anonymous polycephalic author; at random the name Bourbaki, a not well known French maréchal at the times of Napolean III, was proposed.

Later a first initial had to be attributed to Bourbaki. It became N., probably for Nicolas.

As first activities were more and more located in Nancy and one of the most influential members A. Weil was staying in Chicago, the address adopted was the imaginary city of Nancago.

Later collaborators, among others, will be Claude Chabauty, Jacques Dixmier, Roger Godement, Alexander Grothendieck, Jean - Louis Koszul, Georges de Rahm, Pierre Samuel, Laurent Schwartz, Jean - Pierre Serre, John Tate, and René Thom.

Up to 1983 some forty issues have been published. Including reeditions, renewed and enriched versions, but not translations, one should add some thirty supplementary issues.

The table of contents of *Éléments de mathématique* contains a first part describing all kinds of structures, a common ad hoc ground for a coherent program with increasing complexification.

The term Elements of course does not mean elementary. The guideline is the choice of a powerful system of structures adequate for comprehensive presentations able to provide a kit of mathematical tools. It is a codifying enterprise endowed with constructive possibilities that should contribute to enhancing growth of mathematics.

Laurent Schwartz, the author of distribution theory, credits himself as the most algebraized analyst.

Actually, first algebraic axioms, then topological axioms and finally combined structures are considered. The numerical line with its rich structure comes up later.

In the second part the studied topics are not ranged in a specific order.

The adopted rule is going from simple situations to complicated ones, from the general case to particular instances. That principle is guided by Hilbert's *Axiomatisches Denken*. Bourbaki endeavors striving towards the maximum of useful generality.

3. The Impact

In 1962 Bourbaki explains his views on the *Architecture of mathematics*:

"Today […] we think that the internal evolution of the mathematical science has, albeit its appearances, reinstalled more than ever the unity of its various parts, and has created some sort of central kernel, more coherent than it has ever been. The essence of this evolution consisted in a systematization of the relations existing between the various mathematical theories, and is summarized in a tendency commonly known by its name 'axiomatic method' (1962, p. 37).

Denying any form of formalist deviationism Bourbaki adds:

"It is the outside form that the mathematician attributes to his thought, the motor which makes it accessible and, in order to be complete, the proper language of mathematics; but one should not try to find more therein" (loc. cit.).

"Also every mathematician knows fairly well that a demonstration is not really 'understood' as long as one has just checked step by step all the deductions it contains, without trying to conceive clearly the ideas having led to that specific chain of deductions instead of any other" (loc. cit.).

Bourbaki is well aware of the real motor of the evolution of mathematics:

"More then ever intuition is the major force in the genesis of discoveries; but it now possesses strong levers produced by the great variety of structures, and it dominates in a single eye view immense domains unified by axiomatics, where in former times it seemed to be ruled by the most unorganized chaos" (1962, p. 43).

Christian Houzel further explains Bourbaki's motivations:

"For each theory one should analyze demonstrations in order to detect the used hypotheses; these hypotheses adopted as axioms afford a theory lacking unnecessary hypotheses and revealing itself as applicable to a much wider domain. The fertility of the method is due to the fact that ideas relevant for one application of the theory may be transferred to the general case and so enrich other theories" (1999, 25).

Bourbaki gave instructions to the reader. His views were of course much too optimistic. In the introduction of each volume one reads:

"In principle, no specific mathematical knowledge is required, but simply a certain habit of mathematical reasoning and a certain acquaintance with abstraction."

In contrast the *Elements of History of Mathematics* adopt a colloquial pleasant style in order to explain the chronology of modern mathematical subjects.

Quite often Bourbaki is accused of indigestive presentations. He was after the most rigorous style, compatible with general situations.

Yet, one could have preferred sentences like 'Riesz - Fischer theorem' instead of some several digits classification.

Bourbaki himself explains his numerization system by the example: AC. III. §4 N°5 Cor. de la prop. 6.

Bourbaki's choices of axioms are far from being always the best. The most often quoted criticism is about integration theory and measure theory not easily transferable to probability theory. The topologies adopted on EVT are not the most natural ones. Also it would have been easier to suppose existence of a unit in any ring.

The absence of considerations of mathematical logic was regretted.
Bourbaki was charged to evacuate all kinds of mathematical intuition. Jean Dieudonné formulated his defence:

"Progresses produced by intuition [in contrast to what one would think] are produced parallel to the development of abstraction. The more abstract things are, the more they favor intuition. Why? Because abstraction eliminates whatever is contingent in a theory" (1981, I, p. 21).

André Lichnerowicz, who had never been a member of Bourbaki, makes a general observation:

"Mathematics is a science of concrete objects, because abstraction intends to be a concrete action of our mind."

As mathematics is abstract, it is polyconcrete, a multipurpose knowledge.

Bourbaki was often reproached some kind of dogmatism, but one should know about Bourbaki's limited aims.

In a letter sent to Henri Cartan by Jean Dieudonné on December 2, 1946 one reads:

"About the extension of an operator ring to the noncommutative module, I think by making a reference in a footnote we make clear that we were aware of that questioning,

and we definitely cannot foresee nonexisting theories: we are writing the Bourbaki of 1947, not the Bourbaki of 2000"*.

The symbol * was added in that letter by André Weil who did prefer ≤ 1960.

Of course nowadays, as geometry reenters the scenery in force, Henri Poincaré's heritage may be more acknowledged than the axiomatic rigor.

Bourbaki has been attacked from different sides.

Nevertheless there were so called Bourbakists whose style was nearly completely bourbakist, but who had never been a member. Marcel Brelot is a good example.

Arnaud Denjoy testified:

"Impressed by the facility with which Mr. Dieudonné is able to quote Bourbaki by heart and in odder to fix that multiple value in one of its possible determinations, we find it convenient to see Bourbaki in Mr. Dieudonné's faces" (1964, p. 232).

Actually Dieudonné produced a challenging book on *MathŽmatiques bourbachiques* in which he mainly summarized and explained the Bourbaki Seminar lectures approximately up to 1977, grading them from A to D.

In the top class he ranges algebraic and differential topologies about which he declares that they were created in the last years of the XIXth century by Poincaré in order to settle Riemann's intuitions on firm grounds.

It was only in the XVIIth century that a tendency came up claiming mathematics is just producing commonplace tools.

Jean Dieudonné commented:
"If that idea had prevailed, it would doubtlessly have swept away most of the domains conquered by mathematicians since those times and even probably the ones for which today's utilitarians care the most" (1978, I, p. 10-11).

Today very globally, applied mathematics is reevaluated in comparison with so called pure mathematics. And models seem to be more popular items than structures.

Of course Bourbaki's *Opera omnia* should be ranged under 'pure mathematics'.

During the XXth century, many of the very extraordinary results in pure mathematics and proofs of longstanding conjectures solved by Fields Medalists, were achieved with abstract tools manufactured during the Bourbaki epoch.

An extraordinary illustration of an impressive elaborated mathematical knowledge is displayed in the Bourbaki treatise, considered by Saunders MacLane as *"that magnificent multivolume monster […] a splendid formal organization of many* *advanced topics, formulated in blissful disregard of the origins and applications"* (1986, p. 5).

En 1975, Juli A. Schreider writes about real objects and set concepts:

"Mathematicians would now be quite astonished if it turned out that some mathematical object could not be interpreted as a set with a certain structure of relations defined in it. The idea of a group of French mathematicians, having taken the pen-name of N. Bourbaki, that any mathematical object is a set, provided with a certain structure (algebraic or topological), appears to be (and, to a certain extent, is) the highest achievement of mathematical consciousness.

Incidentally, specialists in mathematical logic would hardly agree with such a peremptory judgment. In mathematical logic, the following are considered as objects of investigation: procedures (algorithms, recursive processes, etc.), properties (intentionally given predicates) and formal theories (considered independently of the models - sets with relations - which embody them). None of these objects are directly reducible to structures in the sense of N. Bourbaki.

Nevertheless, the set-theoretical point of view has gained so many adherents, that it has in a certain sense become a universal scientific conception" (1975, p. 251).

Along that subject, let us quote from Yurii I. Manin:

"The crucial distinction between the ways we present our ideas in the last half of this century lies not so much in our attitudes towards a rigorous proof as towards exact definitions"(1998, p. 166).

"In fact, barring mistakes, the most crucial difficulty with checking a proof lies usually in the insufficiency of definitions (or lack thereof)" (loc. cit.).

"The whole ideology of Bourbaki's treatise consists in representation of mathematics as a building supported by a strict system of good definitions (axioms of basic structures). And since a good definition is sometimes the work of generations of good mathematicians, the temptation to believe that we already know them all may be great" (op. cit., 166-167).

Bourbaki was very keen on the introduction of new terms, excluding for instance words accumulating Latin and Greek origins. Unimorphism was to be replaced by monomorphism.

Bourbaki distinguishes between a ball and a sphere. He also introduces a distinction between *revêtement* and *recouvrement*.

For Christian Houzel:
"It may happen that the Bourbaki style had the broadest influence on the later mathematics" (1999, 7).
Bourbakists acknowledge:

“We do not claim to rule for all eternity. Maybe once upon a time mathematicians will agree to use a mode of reasoning that cannot be formalized in the present language; then it will be necessary if not to change the whole language at least to increase the rules of syntax. The future will have to decide!” (1970, 1.9).

But Bourbaki is not just respectable monument of the past. It remains influential as a fruitful source of concepts and references.

A side-aspect is the institutional *Séminaire Bourbaki* taking place during three weekends each year at the Institut Henri Poincaré in Paris, since 1948.

The program features conferences, most often up to six, by specialists on recent mathematical achievements, generally obtained by somebody else, exceptionally by themselves.

The first session, December 1948, was opened by Henri Cartan with the title *Les travaux de Koszul I*. During the same session another talk was given by Roger Godement.

All conferences are still available in print.

Bourbaki was also made responsible for the ‘failure’ of ‘modern mathematics’.

There have been shortcomings indeed. Yet Bourbaki never favored mere extrapolation of his methods to mathematics teaching.

Yet, if initially the Bourbaki pioneers had intended to write textbooks, later one could only justify the outcome by the fact that each major subsection features a list of exercises, for which the proofs are very tough in general.

After 1940, in particular after the publication of the achievements performed by Salomom Eilenberg and Saunders MaxLane (1942), it became clear that it could be profitable to consider the more general framework of categories and functors.

Nevertheless Bourbaki did not embark on a reformulation of his writings that would have led to the still more abstract language of categories. To him it seemed that the knowledge based on set theory is satisfactory, while estimating that the question of the foundations of mathematics is not sufficiently attractive and motivating, due to Gödels important results.

Yves Meyer quotes from Paul-André Meyer:

“As it is fashionable today to discredit Bourbaki I would like to evoke how much I got amazed when I was reading the wonderful first volumes on General Topology” (2005, p. 8).
Bourbaki had at least strong indirect influences on mathematics as a whole.

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Biographical Sketch

Jean - Paul Pier, mathematician, is professor. em. of the Luxembourg Higher Education institution. He is the author of several books on mathematics and history of mathematics: *Amenable Locally Compact Groups* (Wiley, New York, 1984) *Amenable Banach Algebras* (Longman, UK, 1988) *L'analyse harmonique,son développement historique* (Masson, Paris, 1990) *Histoire de l'intégration. Vingt-cinq siècles de mathématiques* (Masson, Pars, 1996) *Mathematical Analysis during the XXth century* (Oxford University Press, 2001) *Mathématiques entre savoir et connaissance* (Vuibert, Paris, 2006) He is also the editor of collective volumes, among them *Development of Mathematics 1900-1950* and *-2000* (Birkhäuser, Basel, 1994, 2000). The two books were a prelude to the World Mathematical Year 2000 (WMY2000), unanimously proclaimed by UNESCO, after the proposal initiated by the author had been introduced by the Luxembourg UNESCO Commission.