HISTORY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION

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Contents

Summary

Since longtime mathematics educators have shown interest in the use of history of mathematics in mathematics teaching. In many countries curricula mention the need of introducing a historical dimension in mathematics teaching. This chapter discusses some interesting reasons put forwards by the supporters of this use and their epistemological assumptions.

The initial part of the chapter provides a short account of the setting in which the first discussions and the first experiments concerning the use of history in mathematics teaching took place. Afterwards the chapter outlines the development of the community of scholars interested in the relationship between history and pedagogy of mathematics,

which in 1976 was officially established as the group HPM (History and Pedagogy of Mathematics) affiliated to ICMI (International Commission on Mathematical Instruction). Some materials produced in this context are reported in the Appendix. They constitute a background and a source for researchers and for mathematics teachers wishing to explore the opportunity offered by history in their teaching.

About the introduction of history of mathematics in the classroom the chapter focuses on two main streams of action: - history for promoting the image of mathematics as a vivid discipline with links with reality, - history for dealing with mathematical concepts.

An efficient introduction of history in teaching entails adequate teachers' historical knowledge. Then a part of this chapter is dedicated to discuss the role of this knowledge and to present how history may be used in teacher training programs.

At the end of the chapter some frequent objections put forwards by teachers about the possibility of introducing history in their teaching are presented. The conclusion is that, tough there are difficulties and some contexts are not favorable to this introduction, in suitable contexts the effort required for facing this endeavor will be rewarded by significant improvements in the classroom life.

1. Introduction

Many educators recognize that history of mathematics may have a role in mathematics education. The arguments that support this opinion are various. It is claimed that using history makes mathematics be perceived as a human endeavor, that it allows seeing the multiple facets of concepts and theories, and highlighting obstacles met in mathematical understanding. Moreover, together with epistemology, history of mathematics is considered suitable for setting mathematical objects in specific problematic contexts: evolution of rigor, ideologies, methods, forms of discourse, and links with other disciplines.

In this chapter the role of history of mathematics in mathematics education is discussed through theoretical considerations and a few examples of practice in the classroom and in teacher training. A brief historical survey introduces to the theme by showing that since longtime history of mathematics in mathematics education is a theme that has interested mathematicians and educators.

2. The pioneer Period in the Introduction of History in Mathematics Education

2.1. The Scenario

In the second half of nineteenth century, when old states were modernized and new ones were established, one of the main concerns of the governments was to update or create systems of education in their countries. To this aim all school levels were considered with different objectives and approaches. For the primary level the main problem was the literacy of the population, for more advanced grades it was that of deepening the students' background and to prepare for professions. In the meanwhile mathematical research was developing in many domains.

As always, mathematics was a main topic in curricula and very soon the problem of improving mathematics education became a theme of discussion. At the beginning this discussion was mainly carried out by mathematicians. Slowly mathematics teachers, who were acquiring a defined professionalization, entered the discussion and, in some cases, experienced some innovations in their classrooms. This mainly happened at the end of nineteenth century, when associations of mathematics teachers, journals addressed to mathematics teaching, new mathematics textbooks appeared in various countries, see (Furinghetti, to appear). Later on cooperation in mathematics education developed at an international level thanks to the creation in 1908 of the "International Commission on the Teaching of Mathematics" which may be considered the parent of the present ICMI (International Commission on Mathematical Instruction), see (Furinghetti & Giacardi, 2008).

Among the means considered for enhancing mathematics education there was history of mathematics. Since the eighteenth century important treatises on history of mathematics had been published and later some treatises, such as *A Short account of the history of mathematics* by Walter William Rouse Ball published in London (1888), put at disposal of a large audience historical knowledge,. Journals devoted to history of mathematics were founded in the second half of nineteenth century: the Italian *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche* (founded in 1868 by Baldassarre Boncompagni), the German *Abhandlungen zur Geschichte der Mathematik* (published in 1877 by Moritz Cantor), and the Swedish *Bibliotheca Mathematica* (founded in 1884 by Gustaf Hjalmar Eneström, see Lorey, 1926). Chapter 1 of the ICMI Study volume on "History in mathematics education" reports passages, taken from different epochs and countries, that show the widespread concern about the value of history of mathematics in the mathematical culture, see (Fasanelli et al. 2000). The oldest quotation, dating back to the 1790s, is authored by the outstanding mathematician Joseph Louis Lagrange, who stresses the importance of history for mathematical researchers.

2.2. Pioneer Reflections on the Use of History in Mathematics Education

The development of interest in history of mathematics was soon paralleled by the feeling that knowledge on history of mathematics may have a role in the teaching and learning of mathematics. A document that may be considered an archetype of the discussion about the use of history in mathematics education is the text of the talk delivered by G. Heppel (1893) at the *Association for the Improvement of Geometrical Teaching*, the association parent of *Mathematical Association*, the British association of mathematics teachers. In his article Heppel recalls that historical information appears in treatises of mathematics recently appeared, that some teachers use historical illustrations and that he is using history with private pupils. Before explaining the benefits of the use of history in mathematics teaching, he lists the most important restrictions in this use:

I. The History of Mathematics should not form a separate subject of education, but be strictly auxiliary and subordinate to Mathematical teaching.
II. Only those portions should be dealt with which are of real assistance to the learner.
III. It is not to be made a subject of examination. (Heppel, 1893, pp. 19-20)

To answer the main question "in what ways History makes mathematical study easier, clearer, or more interesting" (p. 22) first of all Heppel remarks that history "gives us stereoscopic views instead of pictures and diagrams. A particular subject may be looked at from many sides, each aspect suggesting a different mode of treatment." (p. 22) Afterwards he considers that through history some row ideas of a concept that has been covered by the successive developments may be highlighted to help the full understanding of this concept. Another benefit is that history contrasts the common feeling that mathematics is a dry subject, by recovering the cultural value of mathematics: "Mathematics is full of life and interest, that it appeals to the imagination as well as to the intellect, that is it has a poetry peculiarly its own." (p. 24) Moreover history of mathematics shows how progress in mathematics "has gone on in answer to the needs that men have felt" (p. 24). As shown in the following, the arguments mentioned by Heppel (1893) are present in the discussion on history in mathematics education until our days.

Heppel's address mainly refers to pupils of primary and secondary levels. Other authors of that period consider also advanced levels. In the title of a paper published in 1899 by the Italian historian of mathematics Gino Loria, history of mathematics is seen as a "coupling link" between secondary and university teaching because it may help to revisit mathematics from an advanced standpoint, see (Furinghetti, 2000). Considering tertiary level involves, in particular, considering mathematics teacher education. In this concern Florian Cajori (1894) wrote in the introduction of his seminal book *A history of mathematics*: "Another reason for the desirability of historical study is the value of historical knowledge to the teacher of mathematics." (p. 3). Another famous book on history of mathematics, written by Hieronymus Georg Zeuthen, was intended for teachers and proposed that the history of mathematics should be part of teachers' general education, see (Zeuthen, 1902). This opinion was shared by mathematicians and educators. In 1904 the third International Congress of Mathematicians, held in Heidelberg, adopted a motion in which the introduction of a historical component in public education as well as the teaching of history of exact sciences in university courses were advocated, see (Krazer, 1905). The theme of history in teacher education permeates the work of the educator David Eugene Smith: the course for mathematics teachers he held at the Michigan State Normal School in Ypsilanti was strongly based on the historical perspective (Donoghue, 2006) and his famous book *The teaching of elementary mathematics* is imbued with history of mathematics, see (Smith, 1904). Historical sections appear in *Elementary mathematics from an advanced standpoint*, the work where the mathematician Felix Klein presents the mathematical content he considered necessary for mathematics teachers, see (Klein, 1939). In part II (*Geometry*), he writes:

I shall draw attention, more than is usually done ... to the historical development of the science …. I hope, by discussions of this sort, to further, as I like to say, your general mathematical culture: alongside of knowledge of details, as these are supplied by the special lectures, there should be a grasp of subject-matter and of historical relationship [emphases in the original]*"* (II, p. 2).

The previous short outline of the pioneer discussion on the use of history in mathematics education stresses the existence of two domains of action: student

education and teacher education. Though there are obvious links and some theoretical frames are common, I'll treat separately the two domains and I'll provide some specific examples for both.

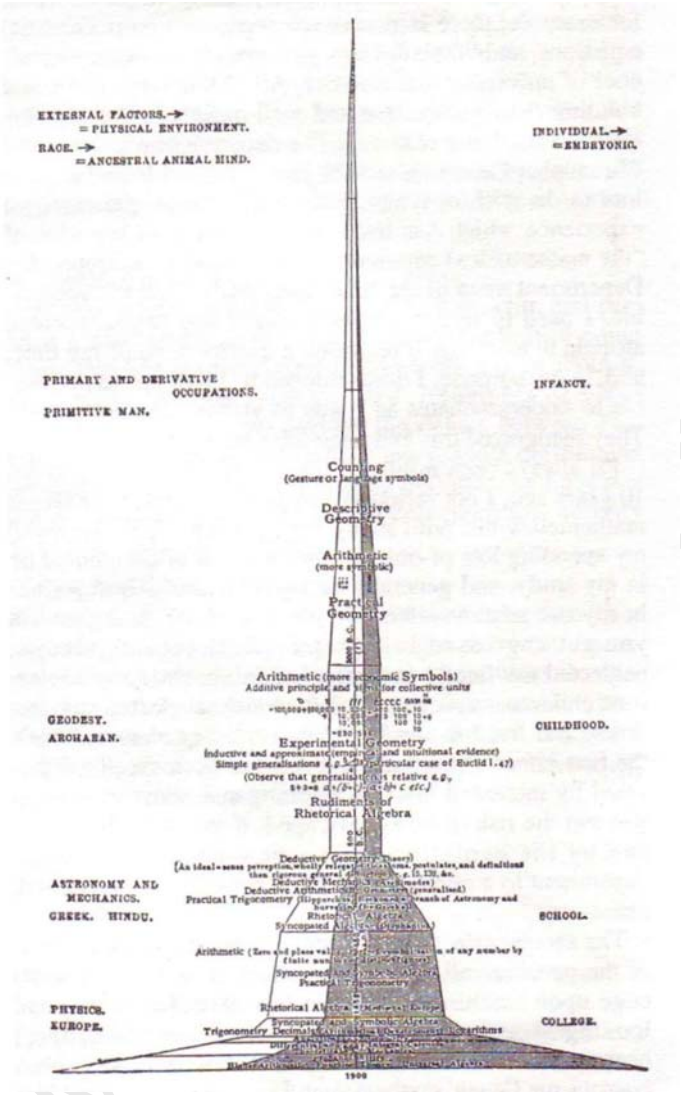

Figure 1. Branford's schema about the mathematical stages of civilizations and of students' education

2.3. A Pioneer Experiment of Introducing History in the Mathematics Classroom

A telling evidence of an early attempt in the use of history is provided by the text of the talk presented at the *Mathematical Association* in 1913 by Miss Barwell, who accounts her introduction of history of mathematics to students of the Training Department of Alexandra College (Dublin) and to girls aged sixteen and seventeen of other classes. The author's very words illustrate the cultural climate and aims of this experiment (Barwell, 1913, p. 72):

While reading for these lectures, I was greatly struck by the stress laid by Benchara Branford in his Study of Mathematical Education *on the importance of what one might*

call the historical method. He emphasises the fact that the history of each individual development is a brief compendium of the history of the race, and that the sound method of instruction is to let the student travel, in his quest for knowledge, roughly over the same path by which his fathers arrived, - roughly, only, because life is short, and there were quagmires in which our fathers floundered for many centuries.

I thought it would be very good for the training-students to learn a little of how "Mathematics" grew, before they studied how to teach them, and so I sacrificed a certain amount of their very limited time to this object. And I was glad to find how much their interest was stimulated - especially among those who knew a little mathematics, - and though it was barely possible to do more than stimulate interest, one hopes that some of them will care enough to read more of the subject for themselves when the brief fever of training is at an end.

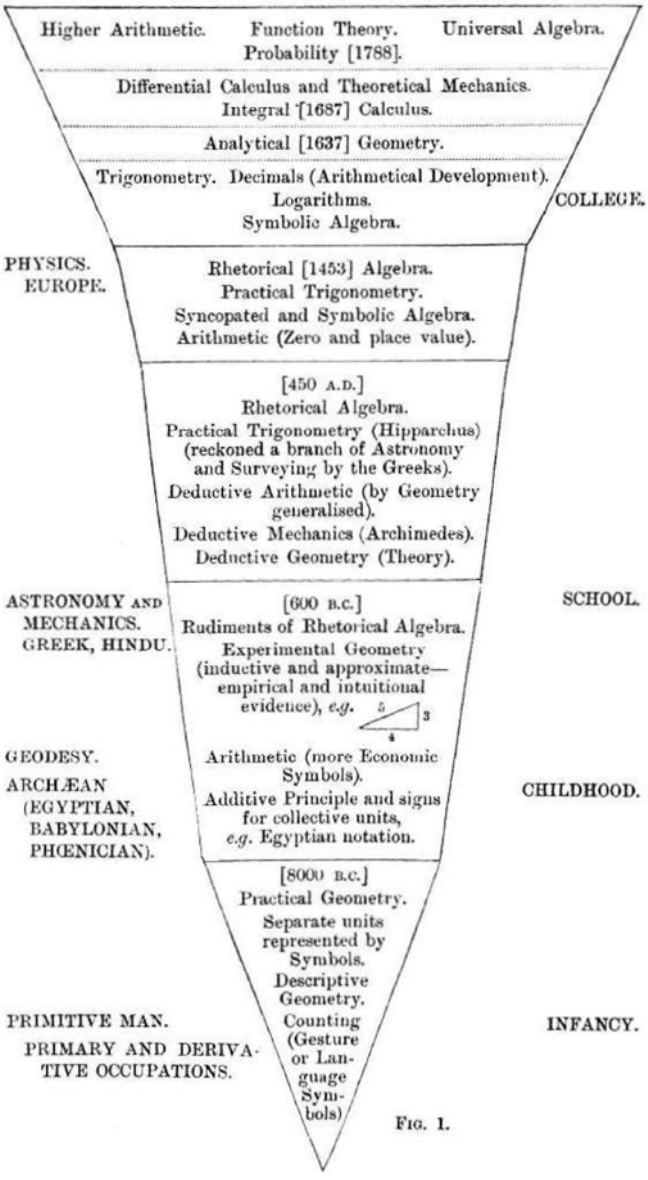

Figure 2. Miss Barwell's adaptation of Branford's schema

In the classroom Miss Barwell treated notations, Egyptian mathematics, systems of numeration, positional notations, the rise of algebra and of geometry. The works she cites in her talk suggest that she was a very special teacher, well acquainted about literature and academic events. The book she mentions in the above quotation is (Branford, 1908), a famous treatise on mathematics teaching that presented many innovative methods of teaching, such as the use of manipulatives and mathematical laboratories, as well as issues outside mathematics such as psycho-analytical theories, in particular the function of sub-consciousness. With this book Branford launched a program of empirical research in mathematics education in years in which there was almost no empirical research. Branford makes often reference to history: his book, in particular, contains a figure outlining the parallelism between mathematical development in the civilizations and individuals' stages of learning, see Figure 1. Branford's use of history has been considered as a form of application of recapitulation (ontogeny recapitulates the phylogeny), see (Fauvel, 1991). Schubring (2006) argues this position and claims "In view of the absence of empirically confirmed propositions concerning the process of learning in mathematics, Branford's approach may be understood as using history of mathematics as a guideline for formulating research questions which then have to be investigated empirically." In (Barwell, 1913) (p. 332) there is an adaptation of the figure conceived by Branford, see Figure 2.

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Biographical Sketch

Fulvia Furinghetti is full professor of Mathematics Education in the Department of Mathematics of the University of Genoa (Italy). Her research concerns mathematics education and history of mathematics education. In the first years of her career she carried out research in projective-differential geometry.She developed projects on the use of history of mathematics in teaching, the history of mathematics education, beliefs, the public image of mathematics, proof and problem solving, algebra, technology in mathematics education, teacher professional development, teachers' and student's beliefs. She has organized the Symposia celebrating the Centenary of *L'Enseignement Mathématique* (Geneva, 2000) and of ICMI (Rome, 2008) and edited the proceedings of both Symposia. She is the author (with Livia Giacardi) of the website on the first hundred years of ICMI http : / / w w w . icmihistory . unito . I t / . In 2000-2004 she chaired HPM, the International Study Group on the relations between History and Pedagogy of Mathematics affiliated to ICMI (International Commission on Mathematical Instruction).