FINANCIAL MARKETS

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Contents

1. Introduction

Since the 1960s, the theory of financial markets has become a growing field of interest for academics as well as for practitioners. We present here an overview of the main topics.

The key concept of the theory is that of "absence of arbitrage". An arbitrage is a way of making money with no initial investment. A minimal requirement of the theory of financial markets is that there should be no-arbitrage. We shall see the use and the limits of the concept in several fields as applied in pricing of derivatives, in consumption-investment problems and lastly to study the term structure.

The survey is organized as follows: In the first section, we focus on the problem of pricing and hedging derivatives. Indeed since the opening of the Chicago board of option exchange in 1973 and the use of Black and Scholes' formula, this topic has probably become the most important in finance given the large variety of financial instruments that have been created. In the second section, we consider next consumption-investment problems, which were studied at a very early stage in the literature. A third section is devoted to financial markets equilibria, in other words to the study of the relations between asset prices and macroeconomics variables. A concise survey of recently developed models for short-term interest rate and zero-coupon dynamics is given in the last section.

In each of the sections, we present discrete time and continuous time models. We end this survey with a bibliography of recent books which deal with the subject.

2. Pricing and Hedging

Assume that a family of underlying assets is given on a time horizon $[0,T]$. We shall first focus on the problem of pricing and hedging derivative products. A derivative security is a security whose value depends on the value of the basic underlying variables. The "price" of the derivative is the amount of money that the buyer agrees to give to the seller of the derivative at time 0 to receive the derivative at date T (the maturity time). When the derivative product is redundant in the market, we shall see that it has a unique fair price, that of a portfolio of underlying assets which gives the same cash flow. Otherwise any investor could achieve a return with no initial investment. When it is not redundant, it may be given several prices. The "hedging strategy" is the portfolio of underlying assets needed by the seller of the derivative to hedge himself against the delivery of the product.

2.1. Discrete Time

2.1.1. Binomial Approach

The simplest example is the so-called two dates "binomial model"'. There are two trading dates, 0 and 1, and two assets : a bond, with price 1 at time 0 and $(1+r)$ at time 1 (r is the interest rate for the period). The asset price equals S at time 0 and is a random variable S_1 at date 1, equal to uS with probability p and to dS with probability $1-p$, with $d<u$. In this simple model, a derivative product is any financial product with payoff C_1 at date 1 equal to $h(S_1)$ for some function h. Examples of derivative products are "call options" and "put options". A call option of strike K has payoff $C_1=(S_1-K)^+$ (while a put option of strike K has payoff $P_1=(K-S_1)^+$) . Indeed one unit of a call option (bought at date 0) confers to the buyer the right (but not the obligation) to buy the asset at price K at date 1. At date 1, if $S_1>K$, the buyer of the call buys the asset at price K and sells it right away at price S_1 making profit S_1-K while if $S_1<K$, the buyer doesn't do anything and his profit is 0. The seller has to deliver the asset at price K if $S_1>K$, this is why he needs to hedge himself against this potential loss.

Let (α,θ) be a portfolio of α shares of bond and θ shares of asset. There are no constraints on α and θ, these numbers can be negative. If θ is negative, the investor is "short" in the asset. The portfolio hedges the derivative if it has same value at time 1, hence if $\alpha(1+r)+\theta S_1=h(S_1)$ equivalently if the following two equations are fulfilled

$$\begin{cases} C_u := h(uS) = \alpha(1+r)+\theta uS \\ C_d := h(dS) = \alpha(1+r)+\theta dS. \end{cases}$$

The hedging portfolio is given by

$$\alpha = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d}\ ,\theta = \frac{C_u - C_d}{S(u-d)}\ ,$$

and the time-0 value of the hedging portfolio can be written as

$$C := \alpha + \theta S = \frac{1}{1+r}\left(\pi C_u + (1-\pi)C_d\right), \qquad (1)$$

where

$$\pi := \frac{1}{u-d}\left((1+r)-d\right). \qquad (2)$$

The right member of (1) can be interpreted as an expectation if π belongs to [0, 1], which is verified if $d<1+r<u$ The number π is then called the "risk neutral probability" since

$$S = \frac{\pi u S + (1-\pi) b S}{1+r}, \tag{3}$$

in other words, the value of the asset is equal to the expectation of its discounted payoff. Equality (1) proves that the value of any portfolio is equal to the expectation of its discounted payoff. If the inequality $d < 1+r < u$ is not fulfilled, then there are ways to make money with no initial investment. More precisely, an arbitrage opportunity is a portfolio (α, θ) such that the initial value is nonpositive $\alpha + \theta S \leq 0$ and the date 1 value is nonnegative

$$\alpha(1+r) + \theta\, uS \geq 0, \alpha(1+r) + \theta\, dS \geq 0$$

and at least one of the three inequalities is strict. For example, let us show that if $(1+r) < d$, then there is an arbitrage. Indeed, at date 0, an investor may borrow the amount of money S at interest rate r and with the money, buy the asset. At date 1, he reimburses $S(1+r)$ and sells the asset at price $S_1 \geq dS$ making the nonnegative net profit $S_1 - S(1+r)$ in both states and a strictly positive profit in the up state. Hence the portfolio $(-S,1)$ is an arbitrage. Symmetrically, in the case $u < (1+r)$, the portfolio $(S,-1)$ is an arbitrage (the investor shorts the asset). Hence if there is no-arbitrage, $d < 1+r < u$ and the price of the asset is its expected discounted payoff under the risk neutral probability. Similarly the value of any portfolio is its discounted expected payoff under the risk neutral probability. It can be proved that if the price of the derivative was different from C, as defined in (1), then there would exist arbitrage opportunities.

2.1.2. Two Dates, Several Assets and Several States of the World

We now consider a two dates financial market where uncertainty is represented by a finite set of states $\{1,\ldots,k\}$. There are $d+1$ assets. At date 0, asset i, $0 \leq i \leq d$, has value S^i and pays $d^i(j)$ at date 1, in units of accounts, in state j. Let $d^i \in \mathbb{R}^k$ be asset's i payoff vector. Assume that asset 0 is riskless (in other words that $d^0(j) = 1, \forall j$) and let interest rate r be defined by $S^0 = \frac{1}{1+r}$. A portfolio $\boldsymbol{\theta} = (\theta^0, \theta^1, \ldots, \theta^d)$ where $\theta^i \in \mathbb{R}$ is the fraction of asset i held by an investor, has market value $\sum_{i=0}^{d} \theta^i S^i = S \bullet \theta = 0$ at date 0 payoff $\sum_{i=0}^{d} \theta^i d^i(j)$ at date 1 in state j. Let $\mathbf{S} \in \mathbb{R}^{d+1}$ be the date 0 assets market values vector and $\mathbf{D}$ be the $(k \times (d+1))$ matrix of payoffs. There is no-arbitrage if $\mathbf{D}\theta = 0$ implies $S \bullet \theta = 0$ and $D\theta \geq 0$, $D\theta \neq 0$ implies $S \bullet \theta > 0$. In other words, there is no-arbitrage if there is no portfolio offering something for nothing. It follows from a convex analysis type of argument that there is no-arbitrage if there exists a “state price” vector $\beta \in \Re^k_{++}$ such that

$$S^i = \sum_{j=1}^{k} d^i(j)\beta_j \quad i \in \{0,...,d\}.$$

One can think of β_j as the cost of obtaining one unit of account is state j. As

$S^0 = \frac{1}{1+r} = \sum_{j=1}^{k} \beta_j$, define $\pi_j = (1+r)\beta_j$. Then $\sum_{j=1}^{k} \pi_j = 1$. We now have a vector of probabilities (such a probability is called "risk neutral") and can write

$$S^i = \frac{1}{1+r} = \sum_{j=1}^{k} \pi_j d^i(j) \qquad i \in \{0,...,d\}.$$

Hence if there is no-arbitrage, the price of an asset is its expected discounted payoff under a well chosen probability.

A contingent claim is a date 1 random payoff and is identified to a element of $\mathbb{R}^k$. Markets are complete if span $D = \Re^k$. In other words, any contingent claim $z \in \Re^k$ may then be hedged (for any $z \in \Re^k$, there exists a portfolio θ such that z = Dθ). If follows from elementary algebra that β and (π, r) are uniquely defined. The date 0 value of a contingent claim z is the initial value of any hedging portfolio θ and is equal to

$$\theta \bullet S = \frac{1}{1+r} \sum_{j=1}^{k} \pi_j z_j = \sum_{j=1}^{k} \beta_j z_j$$

its payoff value at state price β or to its expected payoff under the risk neutral probability. One easily shows that this is the only fair price of the contingent claim: if the contingent claim was given any other price, then there would exist an arbitrage.
If markets are incomplete, one can similarly price any contingent claim z in span D by the value of any hedging portfolio. If z ∉ span D, one cannot price z by arbitrage, one can only define a "bid-ask" spread. Let

$$\overline{S}(z) := \mathit{inf}\left\{\theta \bullet S \middle| D\theta \geq z\right\}$$

be the minimum expenditure of the seller of the contingent claim and

$$\underline{S}(z) := \mathit{sup}\left\{\theta \bullet S \middle| D\theta \leq z\right\}$$

be the maximal amount of money that the buyer of z can borrow against z. Any price in $\left[\underline{S}(z), \overline{S}(z)\right]$ is a no-arbitrage price. Furthermore

$$\overline{S}(z) = \mathit{sup}\left\{\frac{\pi^T z}{1+r} \gg 0,\ \frac{D^T \pi}{1+r} = S\right\} \text{ and } \underline{S}(z) = \mathit{inf}\left\{\frac{\pi^T z}{1+r} \middle| \pi \gg 0,\ \frac{D^T \pi}{1+r} = S\right\}.$$

Similarly if there are portfolio constraints or transactions costs, one may only define a "bid-ask" spread. For example, assume that investors bear the constraints $\theta^{\ell} \geq 0, 0 \leq \ell \leq d_0$. Then the definition of no-arbitrage has to be changed: there is no-arbitrage in the market if there is no feasible portfolio that gives something for nothing (in other words, there is no-arbitrage if $\theta^{\ell} \geq 0, 0 \leq \ell \leq d_0$, $D\theta \geq 0$, implies $S \bullet \theta > 0$). If one defines

$\overline{S}(z) := inf\left\{\theta \bullet S \middle| \theta^{\ell} \geq 0, 0 \leq \ell \leq d_0, D\theta \geq z\right\}$ and $\underline{S}(z) := sup\left\{\theta \bullet S \middle| \theta^{\ell} \geq 0, 0 \leq \ell \leq d_0, D\theta \leq z\right\}$, any price in $\left[\underline{S}(z), \overline{S}(z)\right]$ is a no-arbitrage price.

2.1.3. Multiperiod Discrete Time Model

Let us now study the case of N trading dates.

Let us first assume that there are only two assets, a riskless and a risky asset. The riskless asset has price $(1 + r)^n$ at date n (we assume here that the interest rate is constant over time and denote by $R_n = (1 + r)$-n the time n discount factor) while the risky asset has price S_n. Let us assume that the investor observes past prices and make decisions that depend only on those observations. To model that assumption, we associate to the investor's information a tree. We shall consider that at time 1, there are two states u and d; state u in term is followed by states uu and ud at date 2, the state uu is followed by uuu and uud and so on. A state of nature at time n is a sequence of length n of digits u and d; if e_n is such a sequence, the following states of nature at time n + 1 are denoted by (e_n, u) and (e_n, d). Let $S_n(e_n)$ be the value of the asset at time n in state e_n. A portfolio $(\alpha_n(e_n), \theta_n(e_n))$ held at time n in state e_n, has value $\alpha_n(e_n)(1 + r)n + \theta_n(e_n)S_n(e_n)$ in that state and value $\alpha_n(e_n)(1 + r)^{n+1} + \theta_n(e_n)S_{n+1}(e_n, u)$ or $\alpha_n(e_n)(1 + r)^{n+1} + \theta_n(e_n)S_{n+1}(e_n, d)$ at date n + 1. At date n + 1, the investor may rebalance his portfolio under a "self-financing" constraint: $(\alpha_{n+1}, \theta_{n+1})$ has to fulfill at date n+1 in state e_{n+1},

$$\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n)S_{n+1}(e_{n+1}) = \alpha_{n+1}(e_{n+1})(1+r)^{n+1} + \theta_{n+1}(e_{n+1})S_{n+1}(e_{n+1}).$$

In other words, the value at date n + 1 of the portfolio bought at date n equals the value at date n + 1 of the portfolio bought at date n + 1. When the market is arbitrage free between succeeding states of nature, one may construct, as in (2), node by node a probability on the tree, such that the discounted asset price process is a martingale. More precisely, for any n and any e_n, we introduce two nonnegative numbers $\pi_n(e_n; u)$ and $\pi_n(e_n; d)$ such that $\pi_n(e_n; u) + \pi_n(e_n; d) = 1$ and (cf. (3))

$$S_n(e_n)(1+r) = \pi_n(e_n; u)S_{n+1}(e_n; u) + \pi_n(e_n; d)S_{n+1}(e_n; d).$$

In an explicit form

$$\pi_n(e_n; u) = \frac{(1+r)S(e_n) - S(e_n, d)}{S_n(e_n, u) - S(e_n, d)}$$

represents the risk-neutral probability between time n and n + 1 for the branch of the tree starting at the node e_n. The discounted value of any self-financing portfolio is then also a martingale. Furthermore, one may compute the time n value of a terminal payoff C_N and a hedging strategy by a backward induction argument. Indeed, the N - 1 time value in state e_{N-1} of payoff C_N is (cf. (1)).

$$C_{N-1}(e_{N-1}) = R_N \sum_{s_N} \pi_{N-1}(e_{N-1}; s_N)\, C_N(e_{N-1}, s_N)$$

where s_N = u or d. Similarly the N-2 time value in state e_{N-2} of payoff C_{N-1} is

$$\begin{aligned} C_{N-2}(e_{N-2})R_{N-2} &= R_{N-1} \sum_{s_{N-1}} \pi_{N-2}(e_{N-2}; s_{N-1}) C_{N-1}(e_{N-2}, s_{N-1}) \\ &= R_N \sum_{S_{N-1}, S_N} \pi_{n-2}(e_{N-2}; s_{N-1}) \pi_{N-1}(e_{N-2}, s_{N-1}; s_N) C_N(e_{N-2}, s_{N-1}, s_N) \end{aligned}$$

where s_{N-1} = u or d and $\pi_{N-1}(e_{N-2}, s_{N-1}; s_N) = \pi_{N-1}(e_{N-1}; s_N)$ is the risk neutral probability at node $(e_{N-2}, s_{N-1}) = e_{N-1}$ between time N – 1 and N. The time n-value of payoff C_N in the state e_n = e is therefore obtained by induction

$$C_N(e)R_n = R_N \sum_{s_{n+1}, s_{n+2}, \dots, s_N} \pi_n(e; s_{n+1}) \dots \pi_{N-1}(e, s_{n+1}, \dots, s_{N-1}; s_N)\, C_N(e, s_{n+1}, s_{n+2}, \dots, s_N)$$

where s_i = u or d. The discounted value of time n-value of payoff C_N is therefore the conditional expectation of the discounted terminal value, given the information up to time n, i.e., knowing which states of nature is realized.

Let us now assume that uncertainty is represented by a finite set of states {1,..., k} at each date (for simplicity, we assume that the number of states is constant over time). A state of nature at time n is a sequence of length n of states at dates $\ell \le n$; if e_n is such a sequence, then $e_{n+1} = (e_n, j)$, $j \in \{1,\dots, k\}$. We assume that there are d + 1 assets. At date n, the i-th asset has ex-dividend price S_n^i and pays dividend d_n^i (the cum-dividend price is $S_n^i + d_n^i$). A portfolio (α_n, θ_n) held at time n in state e_n, has value $\alpha_n(e_n)(1+r)^n + \theta_n(e_n) \bullet S_n(e_n)$ in that state at time n and value

$$\begin{aligned} &\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n) \bullet (S_{n+1}(e_{n+1}) + d_{n+1}(e_{n+1})) \\ &= \alpha_n(e_n)(1+r)^{n+1} + \sum_{i=1}^{d} \theta_n^i(e_n) \bullet \left(S_{n+1}^i(e_{n+1}) + d_{n+i}^i(e_{n+1})\right) \end{aligned}$$

at date n + 1. We assume that strategies are "self-financing": the portfolio $(\alpha_{n+1}, \theta_{n+1})$ at date n + 1 in state e_{n+1} has to be such that.

$$\alpha_n(e_n)(1+r)^{n+1}+\theta_n(e_n)\bullet\left[S_{n+1}(e_{n+1})+d_{n+1}(e_{n+1})\right]$$
$$=\alpha_{n+1}(e_{n+1})(1+r)^{n+1}+\theta_{n+1}(e_{n+1})\bullet S_{n+1}(e_{n+1})$$

When the market is arbitrage free between succeeding states of nature (in other words if there doesn't exist strategies such that for a pair (e_n, e_{n+1}), the following inequalities are satisfied $\alpha_n(e_n)(1+r)^n+\theta_n(e_n)\le 0$ while $\alpha_n(e_n)(1+r)^{n+1}+\theta_n(e_n)\left[S_{n+1}(e_{n+1})+d_{n+1}(e_{n+1})\right]\ge 0$ with a strict inequality for some state), one may construct node by node a probability Q on the tree, such that defining

$$\tilde{S}_n^i=\frac{S_n^i}{(1+r)^n},\tilde{d}_n^i=\frac{d_n^i}{(1+r)^n}$$

the discounted price and dividend processes the i-th asset and $\tilde{G}_n^i=\sum_{\ell=1}^{n}\tilde{d}_\ell^i+\tilde{S}_n^i$ the discounted gain, one has

$$\tilde{G}_{n-1}^i=E_Q\left[\tilde{G}_n^i\,|\mathcal{F}_{n\text{-}1}\right].$$

The discounted gain process is therefore a martingale.

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Bibliography

Biais, B., Björk, T., Cvitanić, J., El Karoui, N., Jouini, E. and Rochet, J.C. (1997). *Financial Mathematics, Bressanone*, 1996, Runggaldier, W. ed., volume 1656, Lecture Notes in Maths. Springer-Verlag: Berlin. [Contains a series of paper on Risk sharing, adverse selection and market structure; Interest rate theory; Optimal trading under constraints; Nonlinear pricing theory; Market imperfections, equilibrium and arbitrage.]

Bingham, N.H. and Kiesel, R. (1998). *Risk-Neutral Valuation*, Springer-Verlag: Berlin. [Contains an introduction to incomplete markets, as well as exotic options]

Björk, T. (1998). *Arbitrage Theory in Continuous Time*, Oxford University Press: Oxford. [Deals with change of numeraire, incomplete markets, term structure]

Chung, K. L. and Williams, R. J. (1990). *Introduction to Stochastic Integration*, 2nd edition, Birkhäuser-Verlag [Is a good introduction to stochastic calculus]

Duffie, D. (1992). Dynamic asset pricing theory. Princeton University Press, Princeton, 1992. [Essential]

Elliott, R. and Kopp,P. E. (1998). *Mathematics of Financial Markets*, Springer-Verlag: Berlin. [Deals with discrete and continuous time models. Contains a chapter on American options are studied]

Karatzas, I. and Shreve, S. (1998). *Methods of Mathematical Finance*, Springer-Verlag: Berlin.

Korn, R. (1998). *Optimal portfolio*, World: Singapore. [Contains a detailed presentation of consumption/investment optimization]

Kwok, Y.K. (1998). *Mathematical models of Financial Derivatives*, Springer Finance: Berlin. [Various derivative products are presented]

Lamberton, D. and Lapeyre, B. (1998). *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall. [A good introduction to stochastic calculus, as well as an introduction to option pricing. Contains chapters on American options and processes with jumps]

Mikosch, T. (1999). *Elementary Stochastic calculus with Finance in View*, World Scientific: Singapore. [A concise and excellent introduction to stochastic calculus]

Musiela, M. and Rutkowski, M. (1997). *Martingale Methods in Financial Modeling*, Springer- Verlag: Heidelberg. [A reference book for Financial mathematics, in particular this book contains a study of the term structure and derivative products on interest rates]

Oksendal, B. (1998). *Stochastic Differential Equations*, Fifth edition Springer-Verlag: Berlin. [A complete presentation of stochastic techniques for Brownian motion]

Pliska, S.R. (1997). *Introduction to Mathematical Finance*, Blackwell: Oxford. [Discrete time finance, pricing and optimization techniques]

Shiryaev, A. (1999). *Essential of Stochastic Finance*, World Scientific: Singapore. [A reference book for stochastic finance, in particular for discontinuous models in continuous time]

Wilmott, P., Howison, S. and Dewynne, J. (1995). *The Mathematics of Financial Derivatives. A Student Introduction.*, Cambridge University Press: Cambridge. [PDE approach to exotic options pricing]

Bibliographical Sketches

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