A PERSPECTIVE ON CONSTRUCTIVE QUANTUM FIELD THEORY

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Keywords: constructive quantum field theory, axiomatic quantum field theory, Euclidean quantum field theory, quantum field models, operator algebras, (generalized) stochastic processes, functional integrals, stochastic differential equations

Contents

Summary

The background and motivations of constructive quantum field theory are sketched, and the Wightman and Haag–Araki–Kastler axioms are specified. The goals of constructive and axiomatic quantum field theory are briefly indicated.

The rest of the chapter consists of briefly describing the methods and results of various mathematically rigorous approaches to the construction of quantum field models. This work is roughly organized by method and by chronology.

The chapter closes with an outlook on the status and future of constructive quantum field theory.

1. Introduction: Background and Motivations

Quantum field theory (QFT) is widely viewed as one of the most successful theories in science — it has predicted phenomena before they were observed in nature (for example, the existence and properties of the W and Z bosons, as well as the top and charm quarks, were predicted before they were found experimentally) and its predictions are believed to be confirmed by experiments to within an extraordinary degree of accuracy (for example, the two parts in one billion difference between the theoretical prediction from the Standard Model and the experimentally measured value of the anomalous magnetic moment of the muon).

Though it has undergone a long and complex development from its origins — a certain amount of arbitrariness and personal taste must go into pointing to a single point of origin, since the 1927 discussion of a quantum theory of electromagnetic radiation by Dirac as well as the studies of relativistic wave mechanics by Dirac, Schrödinger and even de Broglie were influential; in any case, the interested reader should see Schweber (1994) for a detailed account of the birth of QFT — in the 1929/30 papers of Heisenberg and Pauli and has attained an ever increasing theoretical sophistication, it is still not clear in which sense the physically central quantum field theories such as quantum electrodynamics (QED), quantum chromodynamics (QCD) and the Standard Model (SM) are mathematically well defined theories based upon fundamental physical principles that go beyond the merely ad hoc. Needless to say, there are many physicists working with quantum field theories for whom the question is of little to no interest. But there are also many who are not satisfied with the conceptual/mathematical state of quantum field theory and have dedicated entire careers to an attempt to attain some clarity in the matter.

This is not the place to explain the grounds for this dissatisfaction; instead, the goal of this chapter is to provide a perspective on "constructive quantum field theory" (CQFT), the subfield of mathematical physics concerned with establishing the existence of concrete models of relativistic quantum field theory in a very precise mathematical sense and then studying their properties from the point of view of both mathematics and physics. Although the insights and techniques won by the constructive quantum field theorists have proven to be useful also in statistical mechanics and many-body physics, these successes of CQFT are not discussed here. In addition, we shall restrict our attention solely to relativistic QFT on d dimensional Minkowski space, $d \geq 2$; to this point, most work in CQFT has been carried out precisely in that context. Throughout, as is customary in QFT, we adopt physical units in which $c = h/2\pi = 1$ where c is the velocity of light and h is Planck's constant..

In the 1950's and early 1960's various "axiomatizations" of QFT were formulated. These can be seen to have two primary goals — (1) to abstract from heuristic QFT the fundamental principles of QFT and to formulate them in a mathematically precise framework; (2) on the basis of this framework, to formulate and solve conceptual and mathematical problems of heuristic QFT in a mathematically rigorous manner. As it turned out, the study and further development of these axiom systems led to unanticipated conceptual and physical breakthroughs and insights, but these are also not our topic here.

The first and most narrow axiomatization scheme of the two briefly discussed here is constituted by the Wightman axioms (see e.g. Streater and Wightman (1964)). This axiom system adheres most closely to heuristic QFT in that the basic objects are local, covariant fields acting on a fixed Hilbert space. A (scalar Bose) Wightman theory is a quadruple $(\phi, \mathcal{H}, U, \Omega)$ consisting of a Hilbert space $\mathcal{H}$, a strongly continuous unitary representation U of the (covering group of the) identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group acting upon $\mathcal{H}$, a unit vector $\Omega \in \mathcal{H}$ which spans the subspace of all vectors in $\mathcal{H}$ left invariant by $U\left(\mathcal{P}_+^\uparrow\right)$ (This condition, referred to as the "uniqueness of the vacuum," is posited for convenience. With known techniques one can decompose a given model into submodels that satisfy this condition as well as the remaining conditions.) and an (unbounded) operator valued distribution (Although it is possible, indeed sometimes necessary, to choose other test function spaces, here we shall restrict our attention to the Schwartz tempered test function space $\mathcal{S}\left(\mathbb{R}^d\right)$.) ϕ such that for every test function f, the operator $\phi(f)$ has a dense invariant domain $\mathcal{D}$ spanned by all products of field operators applied to Ω. These conditions are a rigorous formulation of tacit assumptions made in nearly all heuristic field theories. In addition, a number of fundamental principles were identified and formulated in this framework.

Relativistic Covariance: For every Poincaré element $(\Lambda,a)\in\mathcal{P}_+^{\uparrow}$ one has $U(\Lambda,a)\phi(x)U(\Lambda,a)^{-1}=\phi(\Lambda x+a)$, in the sense of operator valued distributions on $\mathcal{D}$.

Einstein Causality: (Also called microscopic causality, local commutativity or, somewhat misleadingly, locality.) For all spacelike separated $x,y\in\mathbb{R}^4$ one has $\phi(x)\phi(y)=\phi(y)\phi(x)$ in the sense of operator valued distributions on $\mathcal{D}$.

The Spectrum Condition (stability of the field system): Restricting one's attention to the translation subgroup $\mathbb{R}^4\subset\mathcal{P}_+^{\uparrow}$, the spectrum of the self-adjoint generators of the group $U\left(\mathbb{R}^4\right)$ is contained in the closed forward lightcone $\bar{V}_+=\left\{p=(p_0,p_1,p_2,p_3)\in\mathbb{R}^4\,\middle|\,p_0^2-p_2^2-p_3^2\geq 0\right\}$.

The reader is referred to Streater and Wightman (1964), and Jost (1965) for a discussion of the physical interpretation and motivation of these conditions. There is an equivalent formulation of these conditions in terms of the Wightman functions (Streater and Wightman 1964)

$$W_n(x_1,x_2,\ldots,x_n)\equiv\Omega,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\Omega>,n\in\mathbb{N}.$$

which are distributions on $\mathcal{S}\left(\mathbb{R}^{dn}\right)$. These two sets of conditions are referred to collectively as the Wightman axioms. There are closely related sets of conditions for Fermi fields and higher spin Bose fields (Streater and Wightman 1964, Jost 1965).

A more general axiom system which is conceptually closer to the actual operational circumstances of a theory tested by laboratory experiments is constituted by the Haag–Araki–Kastler axioms (HAK axioms), also referred to as local quantum physics or algebraic quantum field theory (AQFT). Although more general formulations of AQFT are available, for the purposes of this paper it will suffice to limit our attention to a quadruple (In point of fact, these conditions actually describe an algebraic QFT in a (Minkowski space) vacuum representation. By no means is AQFT limited to such circumstances; some other representations of physical interest are briefly discussed below. Moreover, the algebraic approach to QFT has proven to be particularly fruitful in addressing conceptual and mathematical problems concerning quantum fields on curved spacetimes.) $\left(\left\{\mathcal{A}(\mathcal{O})\right\}_{\mathcal{O}\in\mathcal{R}},\mathcal{H},U,\Omega\right)$ with $\mathcal{H}$, U and Ω as above and $\left\{\mathcal{A}(\mathcal{O})\right\}_{\mathcal{O}\in\mathcal{R}}$ a net of von Neumann algebras $\mathcal{A}(\mathcal{O})$ acting on $\mathcal{H}$, where $\mathcal{O}$ ranges through a suitable set $\mathcal{R}$ of nonempty open subsets of Minkowski space. The algebra $\mathcal{A}(\mathcal{O})$ is interpreted as the algebra generated by all (bounded) observables measurable in the spacetime region $\mathcal{O}$, so the net $\left\{\mathcal{A}(\mathcal{O})\right\}_{\mathcal{O}\in\mathcal{R}}$ is naturally assumed to satisfy isotony: if $\mathcal{O}_1\subset\mathcal{O}_2$, then one must have $\mathcal{A}(\mathcal{O}_1)\subset\mathcal{A}(\mathcal{O}_2)$. In this framework the basic principles are formulated as follows.

Relativistic Covariance: For every Poincaré element $(\Lambda,a)\in\mathcal{P}_+^\uparrow$ and spacetime region $\mathcal{O}\in\mathcal{R}$ one has $U(\Lambda,a)\mathcal{A}(\mathcal{O})U(\Lambda,a)^{-1}=\mathcal{A}(\Lambda\mathcal{O}+a)$.

Einstein Causality: (Also often referred to as locality.) For all spacelike separated regions $\mathcal{O}_1,\mathcal{O}_2\in\mathcal{R}$ one has $AB=BA$ for all $A\in\mathcal{A}(\mathcal{O}_1)$ and all $B\in\mathcal{A}(\mathcal{O}_2)$.

The Spectrum Condition (stability of the field system): Same as above.

The reader is referred to Haag 1992, Araki (1999) for a discussion of the physical interpretation and motivation of these conditions. The relation between the Wightman axioms and AQFT is well understood. It is important to note that, in general, infinitely many different fields in the sense of the Wightman axioms are associated with the same net of observable algebras. Indeed, an analogy has often been drawn between the choice of a particular coordinate system, made in order to carry out a computation more conveniently, in differential geometry and the choice of a particular field out of the many fields associated with a given net. For this and other reasons, those who work in mathematical QFT consider nets of observable algebras to be more intrinsic than the associated quantum fields, which are used primarily for computational convenience.

Associated to any Wightman system $(\phi,\mathcal{H},U,\Omega)$ is a net of *-algebras $\mathcal{P}(\mathcal{O})$, $\mathcal{O}\subset\mathbb{R}^4$. Because all field operators have the common, dense domain $\mathcal{D}\subset\mathcal{H}$, arbitrary "polynomials" of field operators can be formed on $\mathcal{D}$. $\mathcal{P}(\mathcal{O})$ denotes the algebra formed by all polynomials (in the sense of functions of infinitely many variables) in which the supports of all test functions of all field operators entering into the polynomial are contained in the spacetime region $\mathcal{O}$. The algebras $\mathcal{P}(\mathcal{O})$ are not C*-algebras but satisfy all of the other HAK axioms. Despite the non-intrinsic nature of such algebras and despite the technical disadvantages of working with *-algebras instead of with C*-algebras, mathematical quantum field theorists find it convenient for various purposes to work with such nets or with similar nets of non-C*-algebras.

The goal of constructive QFT, as is it usually understood, is to construct in a mathematically rigorous manner physically relevant quantum field models which satisfy one of these systems of axioms and then to study their mathematical properties with an emphasis on those properties which can be shown to have physical relevance. This article briefly describes such models and the means by which they were constructed and is organized both historically and by the construction techniques employed.

As pointed out independently by Borchers and Uhlmann, the Wightman axioms can be understood in a representation independent manner in terms of what is now called the Borchers (or Borchers–Uhlmann) algebra — a tensor algebra constructed out of the test function space $\mathcal{S}\left(\mathbb{R}^4\right)$ with operations directly motivated by the Wightman axioms. Borchers algebras have been extensively studied from the point of view of QFT, especially by Borchers, Uhlmann, Yngvason and Lassner (see e.g. Horuzhy (1986) for definitions and references). A Wightman system can be thought of as a concrete representation of the Borchers algebra, and for a time there was hope one could arrive at quantum field models by defining suitable states on the Borchers algebra and employing the standard GNS construction to obtain the corresponding representation. However, it proved to be too difficult to conjure such states.

The first quantum field models constructed were the free quantum fields, the Wick powers of such free fields and the so–called generalized free fields. These models have been constructed using a variety of techniques and have been shown to satisfy the two axiom systems discussed above; a recent construction of free fields which is of particular conceptual interest is briefly described in Section 6. The Hilbert space upon which such fields act is called the Fock space. Common to these models is the fact that their S–matrix, the object which describes the scattering behavior of the "particles" described by such fields (cf. Jost 1965, Araki (1999)), is just the identity map.

We turn now to models with nontrivial S–matrices, i.e. interacting quantum field models. When referring to the models, we employ the standard notation M_d, which means quantum model M in d spacetime dimensions. Because the mathematical and conceptual difficulties inherent in the construction of quantum field models are quite daunting, constructive quantum field theorists proceeded by considering increasingly challenging models; this often entailed starting the study of the model M with $d = 2$, then $d = 3$, and finally $d = 4$. At this point in time only a few models have been constructed in four spacetime dimensions. In this respect, the reader is referred to Section 8 for a few words about the outlook for CQFT after nearly fifty years of strenuous effort. The reader should note that all results discussed in this chapter, unless explicitly stated otherwise, are proven according to the criteria accepted by mathematicians and not merely on the basis of the plausibility arguments accepted by most physicists as "proof".

2. Algebraic Constructions I

Preceded by the 1965 dissertations of Jaffe and Lanford, the first constructions of interacting quantum fields were carried out in the late 1960's and early 1970's. In this early work the real time models were constructed directly using operator algebras and functional analysis as the primary tools. Due to Haag's Theorem, it was known that the Hilbert space in which these interacting quantum fields would be defined could not be

Fock space. However, because Fock space was the sole available starting point at that time, “cutoffs” were placed on the interacting theories so that they could be realized on Fock space in a mathematically meaningful manner. These cutoffs were of two general kinds — finite volume cutoffs and ultraviolet cutoffs — each addressing independent sources of the divergences known in QFT since early in its development. Guided by heuristic QFT’s division of Lagrangian quantum field models into superrenormalizable, renormalizable and nonrenormalizable models (this classification is based upon the perturbation theory associated by Feynman and others with interacting fields, viewed as perturbations of free fields), the constructive quantum field theorists began with the simplest category, the superrenormalizable models. To be able to address the infinite volume divergence without wrestling simultaneously with the ultraviolet divergence, constructive quantum field theorists first considered self-interacting bosonic quantum field models in two spacetime dimensions.

We begin with Glimm and Jaffe’s construction of the $\left(\phi^4\right)_2$ model, the self-interacting scalar Bose field on two dimensional Minkowski space with Lagrangian self-interaction $\lambda\phi^4$, where λ is the coupling constant. Let $\mathcal{H}_0$ be the Fock space for a (free) scalar hermitian Bose field $\phi(t,x)$ of mass $m>0$ ($(t,x)\in\mathbb{R}^2$). Let $\pi(t,x)=\partial\phi(t,x)/\partial t$ be the canonically conjugate momentum field and $\mathcal{D}\subset\mathcal{H}_0$ be the dense set of finite-particle vectors in $\mathcal{H}_0$. Then, for every f in a dense subspace $\mathcal{S}(\mathbb{R})$ of $L^2(\mathbb{R})$, the operator $\phi_0(f)\equiv\int\phi(0,x)f(x)dx$ is essentially self-adjoint on $\mathcal{D}$ and $\phi_0(f)\mathcal{D}\subset\mathcal{D}$ (similarly for $\pi_0(f)$). These operators satisfy the canonical commutation relations (CCR) on $\mathcal{D}$:

$$\phi_0(f)\pi_0(g)-\pi_0(g)\phi_0(f)=i<f,g>\mathbb{I},$$

$$\phi_0(f)\phi_0(g)-\phi_0(g)\phi_0(f)=0=\pi_0(f)\pi_0(g)-\pi_0(g)\pi_0(f),$$

for all $f,g\in\mathcal{S}(\mathbb{R})$, where $<\cdot,\cdot>$ is the inner product on $L^2(\mathbb{R})$ and $\mathbb{I}$ is the identity operator on $\mathcal{H}$. When exponentiated using the spectral calculus, (the closures of) these operators provide a Weyl representation of the CCR. For each bounded open subset $\mathbf{O}\subset\mathbb{R}$, denote by $\mathcal{A}(\mathbf{O})$ the von Neumann algebra generated by the Weyl unitaries

$$\left\{e^{i\phi_0(f)},e^{i\pi_0(f)}\,\middle|\,f\in\mathcal{S}(\mathbb{R}),\quad \mathrm{supp}(f)\subset\mathbf{O}\right\},$$

$\mathrm{supp}(f)$ denotes the support of the function f.)

The total energy

$$H_0=\frac{1}{2}\int:\left(\pi(0,x)^2+\nabla\phi(0,x)^2+m^2\phi(0,x)^2\right):dx$$

of the free field is a positive quadratic form on $\mathcal{D}\times\mathcal{D}$ and therefore determines uniquely a positive self-adjoint operator, which we also denote by H_0. The double colons indicate that the expression between them is Wick ordered, which is a physically motivated way to define in a rigorous manner a product of operator valued distributions. In this case, the Wick ordering is performed with respect to the Fock vacuum. With $g\in L^2(\mathbb{R})$ nonnegative of compact support, Glimm and Jaffe showed that, for each $\lambda>0$, the cut-off interacting Hamilton operator

$$H(g)\equiv H_0+\lambda\int:\phi(0,x)^4:g(x)dx$$

is essentially self-adjoint on $\mathcal{D}$ (without the cutoff g, the interacting Hamilton operator is not densely defined in Fock space) and its self-adjoint closure, also denoted by $H(g)$, is bounded from below. By adding a suitable multiple of the identity we may take 0 to be the minimum of its spectrum. Then, they proved that 0 is a simple eigenvalue of $H(g)$ with normalized eigenvector $\Omega(g)\in\mathcal{H}_0$.

For any $t\in\mathbb{R}$, let $\mathbf{O}_t$ denote the subset of $\mathbb{R}$ consisting of all points with distance less than $|t|$ to $\mathbf{O}$. By choosing the cutoff function g to be equal to 1 on $\mathbf{O}_t$, then for any $A\in\mathcal{A}(\mathbf{O})$ the operator

$$\sigma_t(A)\equiv e^{itH(g)}Ae^{-itH(g)}$$

is independent of g and is contained in $\mathcal{A}(\mathbf{O}_t)$. For any bounded open $\mathcal{O}\subset\mathbb{R}^2$ and $t\in\mathbb{R}$, let $\mathbf{O}(t)=\{x\in\mathbb{R}\,|\,(t,x)\in\mathcal{O}\}$ be the time t slice of $\mathcal{O}$. We define $\mathcal{A}(\mathcal{O})$ to be the von Neumann algebra generated by $\bigcup_s\sigma_s(\mathcal{A}(\mathbf{O}(s)))$. One can then show that the algebra $\mathcal{A}(\mathcal{O})$ coincides with the von Neumann algebra generated by bounded functions of the self-adjoint field operators $\int\phi(t,x)f(t,x)dxdt$, with test functions $f(t,x)$ having support in $\mathcal{O}$. Finally, we let $\mathcal{A}$ denote the closure in the operator norm of the union $\bigcup\mathcal{A}(\mathbf{O})$ over all open bounded $\mathcal{O}\subset\mathbb{R}^2$. Hence, σ_t is an automorphism on $\mathcal{A}$ and implements the time evolution associated with the interacting field. Similarly, "locally correct" generators for the Lorentz boosts and the spatial translations can be defined, resulting in an automorphic action α on $\mathcal{A}$ of the entire (identity component of the) Poincaré group $\mathcal{P}_+^\uparrow$ in two spacetime dimensions.

For each $A\in\mathcal{A}$, we set $\omega_g(A)=<\Omega(g),A\Omega(g)>$ ($<\cdot,\cdot>$ denotes here the inner product on $\mathcal{H}$) to define the locally correct vacuum state ω_g of the interacting field. Taking a limit as the cutoff function g approaches the constant function 1, Glimm and

Jaffe showed that $\omega_g(A) \to \omega(A)$, for each $A \in \mathcal{A}$, defines a new (locally normal) state ω on $\mathcal{A}$ which is Poincaré invariant, i.e. $\omega\left(\alpha_{(\Lambda,x)}(A)\right) = \omega(A)$ for all $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ and all $A \in \mathcal{A}$. Employing the GNS construction, one then obtains a new Hilbert space $\mathcal{H}$, a representation ρ of $\mathcal{A}$ as a C^*-algebra acting on $\mathcal{H}$, and a vector $\Omega \in \mathcal{H}$ such that $\rho(\mathcal{A})\Omega$ is dense in $\mathcal{H}$ and

$$\omega(A) = <\Omega, \rho(A)\Omega>, \quad \text{for all } A \in \mathcal{A}$$

In addition, one obtains a strongly continuous unitary representation U of the Poincaré group in two spacetime dimensions under which the algebras $\rho(\mathcal{A}(\mathcal{O}))$ transform covariantly, i.e.

$$U((\Lambda, x))\rho(\mathcal{A}(\mathcal{O}))U((\Lambda, x))^{-1} = \rho(\mathcal{A}(\Lambda\mathcal{O} + x)).$$

Both the HAK and Wightman axioms have been verified for this model.

The generators of the strongly continuous Abelian unitary groups $\left\{\rho\left(e^{it\phi(f)}\right) \middle| t \in \mathbb{R}\right\}$ and $\left\{\rho\left(e^{it\pi(f)}\right) \middle| t \in \mathbb{R}\right\}$ satisfy the CCR. However, this representation of the CCR in $\mathcal{H}$ is not unitarily equivalent to the initial representation in Fock space, in accordance with Haag's Theorem. Indeed, by taking different values of the coupling constant λ in the above construction, one obtains an uncountably infinite family of mutually inequivalent representations of the CCR.

It is in this representation $(\rho, \mathcal{H})$ that the field equations for this model find a mathematically satisfactory interpretation, as shown by Schrader. And it is to the physically significant quantities in this representation that the corresponding perturbation series in λ is asymptotic — see below for further discussion. For this and other reasons, ω is interpreted as the exact vacuum state in the interacting theory corresponding to the Lagrangian interaction $\lambda\phi^4$, and the folium of states associated with this representation contains the physically admissible states of the interacting theory. Many further properties of physical relevance have been proven for this model more recently — see the discussion below in Section 1.

The results attained for the ϕ_2^4 model were subsequently extended by Glimm and Jaffe to $P(\phi)_2$ models (using a periodic box cutoff), where $P(\phi)$ is any polynomial bounded from below. If $P(\phi)$ is not bounded from below, then the corresponding cutoff Hamiltonian $H(g)$ is not bounded from below and the resulting model is not stable. (See Glimm and Jaffe (1987) for more complete references and history of this development.) Hoegh-Krohn employed the techniques of Glimm and Jaffe to construct models in two spacetime dimensions (with similar results) in which the polynomial interaction $P(\phi)$ is replaced by a function of exponential type, the simplest example being $e^{\alpha\phi}$.

Analogous results were proven for Y_2, the Yukawa model in two spacetime dimensions by Glimm and Jaffe and Schrader. In this model one commences with the direct product $\mathcal{H}_0 = \mathcal{H}_\mathrm{b} \otimes \mathcal{H}_\mathrm{f}$ of the Fock space $\mathcal{H}_\mathrm{b}$ for a scalar hermitian Bose field $\phi(t,x)$ of mass $m_\mathrm{b} > 0$ and the Fock space $\mathcal{H}_\mathrm{f}$ for a Fermi field $\psi(t,x)$ of mass $m_\mathrm{f} > 0$. In this model the free Hamiltonian H_0 is the total energy operator of the free fields ϕ and ψ. Because there is still an ultraviolet divergence remaining after Wick ordering, the cutoff interacting Hamiltonian is $H(g,\kappa) \equiv H_0 + H_I(g,\kappa) + c(g,\kappa)$, where $H_I(g,\kappa)$ is the result of applying a certain multiplicative ultraviolet cutoff (which is removed in the limit $\kappa \to \infty$) to the formal expression

$$H_I(g) \equiv \lambda \int g(x)\phi(0,x) :\bar{\psi}\psi: (0,x)\,dx,$$

and $c(g,\kappa)$ is a (finite) renormalization counterterm determined by second-order perturbation theory which diverges as $\kappa \to \infty$ and includes both a mass and vacuum energy renormalization. With both volume and ultraviolet cutoffs in place, $H(g,\kappa)$ is a well defined operator on $\mathcal{H}_0$. Glimm and Jaffe show that as $\kappa \to \infty$ the operator $H(g,\kappa)$ converges in the sense of graphs to a positive self-adjoint operator $H(g)$ with an eigenvector $\Omega(g) \in \mathcal{H}_0$ of lowest energy 0. Once again, they control the limit as $g \to 1$ of the expectations $\omega_g(A)$ for all $A \in \mathcal{A}_\mathrm{b} \otimes \mathcal{A}_\mathrm{f}$ and obtain a state ω on $\mathcal{A}_\mathrm{b} \otimes \mathcal{A}_\mathrm{f}$ that provides a corresponding (GNS) representation of the fully interacting theory. Glimm and Jaffe also prove that the Yukawa field equations are satisfied by the fields in that representation. A similar argument was applied to the $Y_2 + P(\phi)_2$ model by Schrader, where

$$H_I(g) \equiv \lambda \int g(x)\big(\phi(0,x) :\bar{\psi}\psi: (0,x) + :P(\phi):(0,x)\big)dx$$

and $P(\phi)$ is any polynomial bounded from below. The axioms of HAK and Wightman were shown to hold in these models, at least for all sufficiently small values of the coupling constant λ. In addition, by using a mixture of algebraic and Euclidean methods Summers showed that the model manifests further properties of physical relevance, such as the funnel property (also known as the split property) and all assumptions of the Doplicher–Haag–Roberts superselection theory (cf. Araki (1999) and Haag (1992)). Therefore the model also admits the physically expected Poincaré covariant, positive energy, charged representations associated with the generator of the global gauge group of the model, which are mutually unitarily inequivalent.

Along the lines employed in the construction of the Yukawa model in two spacetime dimensions, Glimm and Jaffe also showed for the ϕ_3^4 model that the spatially cutoff Hamiltonian $H(g)$ is densely defined, symmetric and bounded below by a constant $E(g)$ proportional to the volume of the support of g. The renormalization constants in the Hamiltonian $H(g)$ are again given by perturbation theory and involve counterterms to the vacuum energy and the rest mass of a single particle. The proof was technically more challenging than that for Y_2, even though the results were more limited to a significant extent. There was real motivation to find an alternative approach, as described in the next section.

However, before proceeding to the next section we mention the Federbush model, a model of self-interacting fermions in two spacetime dimensions. First proposed by Federbush, the Lagrangian of the model is

$$\sum_{s=\pm 1} \bar{\psi}_s \left(\partial\!\!\!/ - m(s)\right)\psi_s - 2\pi\lambda\epsilon_{\mu\nu} J_1^\mu J_{-1}^\nu ,$$

where $\epsilon_{10} = -\epsilon_{01} = 1$, $\epsilon_{00} = \epsilon_{11} = 0$, $J_s^\mu = \bar{\psi}_s\gamma^\mu\psi_s$ and $m(s) > 0$, $s = \pm 1$. Without cutoffs of any kind, a concrete realization of the Federbush model can be given in terms of certain exponential expressions on a suitable Fock space, and Ruijsenaars proved that this realization satisfies the Wightman axioms when $\lambda \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ (Einstein causality is actually verified only for sufficiently small λ). Of particular interest, he proved that the associated Haag–Ruelle scattering theory is asymptotically complete. The S–matrix is nontrivial, but there is no particle production. The Federbush model was the first non-superrenormalizable model for which any of these properties have been proven.

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Bibliography

[1] S. Albeverio, H. Gottschalk and J.-L. Wu, (1997) Models of local relativistic quantum fields with indefinite metric (in all dimensions), *Commun. Math. Phys*., 184, 509–531. [Details and references for the construction of quantum field models using convolutions of white noise stochastic processes.]

[2] H. Araki, *Mathematical Theory of Quantum Fields* (Oxford University Press, Oxford) 1999. [An account of important aspects of AQFT, including Doplicher–Haag–Roberts superselection theory and scattering theory.]

[3] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann, (1997) SU(N) Quantum Yang–Mills theory in two dimensions: A complete solution, *J. Math. Phys*., 38, 5453–5472. [A mathematically rigorous construction of two-dimensional non-Abelian Yang-Mills theory.]

[4] T. Balaban, (1989) Large field renormalization. II. Localization, exponentiation, and bounds for the R operation, *Commun. Math. Phys.*, 122, 355–392. [Details and references for an incomplete program to construct YM_4.]

[5] H. Baumgärtel and M. Wollenberg, (1992) *Causal Nets of Operator Algebras* (Akademie Verlag, Berlin). [A very mathematical account of aspects of AQFT.]

[6] O. Bratteli and D.W. Robinson, (1981) *Operator Algebras and Quantum Statistical Mechanics*, Volume 1 (Springer Verlag, Berlin) 1979, Volume 2 (Springer Verlag, Berlin). [An account of the techniques of AQFT applied to quantum statistical mechanics; includes a discussion of the mathematics and physics of KMS states and of the Tomita–Takesaki modular theory.]

[7] J. Bros and D. Iagolnitzer, (1988) 2–Particle asymptotic completeness and bound states in weakly coupled quantum field theories, *Commun. Math. Phys*., 119, 331–351. [A mathematically rigorous study of aspects of scattering theory for certain two-dimensional quantum field models.]

[8] R. Brunetti, M. Dütsch and K. Fredenhagen, (2009) Perturbative algebraic quantum field theory and the renormalization groups, *Adv. Theor. Math. Phys*., 13, 1541–1599. [Details and references for the perturbative AQFT program mentioned in Section 7.]

[9] D. Brydges, J. Dimock and T.R. Hurd, (1995) The short distance behavior of $\left(\phi^4\right)_3$, *Commun. Math. Phys*., 172, 143–186. [Details and references for recent developments in techniques to study the $\left(\phi^4\right)_3$ model, including renormalization group techniques.]

[10] D. Buchholz, (1982) The physical state space of quantum electrodynamics, *Commun. Math. Phys*., 85, 49–71. [An important investigation of the physical state space in QED from first principles.]

[11] D. Buchholz, G. Lechner and S.J. Summers, (2011) Warped convolutions, Rieffel deformations and the construction of quantum field theories, *Commun. Math. Phys*., 304, 95–123. [Details and references for the deformation program mentioned in Section 3.]

[12] H. Epstein and V. Glaser, (1973) The role of locality in perturbation theory, Ann. *Inst. Henri Poincaré*, 19, 211–295. [An account of causal perturbation theory.]

[13] P. Federbush, (1990) A Phase cell approach to Yang–Mills theory, V. Analysis of a chunk, *Commun. Math. Phys*., 127, 433–457. [Details and references for an incomplete program to construct YM_4.]

[14] J.S. Feldman, T. Hurd, L. Rosen and J.D. Wright, (1988) *QED: A Proof of Renormalizability*, Lecture Notes in Physics, 312 (Springer-Verlag, Berlin, Heidelberg, New York). [A mathematically rigorous proof that QED is perturbatively renormalizable.]

[15] R. Fernández, J. Fröhlich and A. Sokal, (1992) *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory* , Springer Verlag, New York,. [As described in the title.]

[16] J. Glimm and A. Jaffe, (1987) *Quantum Physics. A Functional Integral Point of View*, Springer Verlag, Berlin, Heidelberg and New York. [An account of the program discussed in Section 3.]

[17] R. Haag, (1992) *Local Quantum Physics*, Springer Verlag, Berlin, Heidelberg and New York,. [An introduction to AQFT.]

[18] S.S. Horuzhy, (1986) *Introduction to Algebraic Quantum Field Theory*, Kluwer Academic Publ., Dordrecht,. [A very brief introduction to algebraic quantum field theory.]

[19] D. Iagolnitzer, (1993) *Scattering in Quantum Field Theories*, Princeton University Press, Princeton, NJ,. [A monograph on mathematically rigorous scattering theory of quantum fields..]

[20] R. Jost, (1965) *General Theory of Quantized Fields*, American Mathematical Society, Providence, RI,. [Classic account of axiomatic QFT, including the Haag–Ruelle scattering theory.]

[21] J. Klauder, (2011) *A Modern Approach to Functional Integration*, Birkhäuser Verlag, New York,. [A monograph containing details and references on the material in Section 4.]

[22] C. Kopper, (2010) On the local Borel transform of perturbation theory, *Commun. Math. Phys*., 295, 669–699. [As described in the title.]

[23] G. Lechner, (2008) Construction of quantum field theories with factorizing S-matrices, *Commun. Math. Phys*., 277, 821–860. [Details and references for the important construction discussed in Section 2.]

[24] V. Rivasseau, (1991) *From Perturbative to Constructive Renormalization,* Princeton University Press, Princeton, NJ,. [A rigorous account of renormalization theory; gives more details on the construction of some of the models discussed in this article.]

[25] D. Schlingemann, (1998) Construction of kink sectors for two-dimensional quantum field theory models: An algebraic approach, *Rev. Math. Phys*., 10, 851–891. [Details and references for the rigorous work on soliton sectors.]

[26] S.S. Schweber, (1994) *QED and the Men Who Made It*, Princeton University Press, Princeton, NJ,. [A history of the beginnings of quantum field theory, which then was largely synonymous with QED.]

[27] E. Seiler, (1982) *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Physics, Vol. 159, Springer-Verlag, Berlin,. [An account of lattice gauge theories and their continuum limits.]

[28] F. A. Smirnov, (1992) *Form Factors in Completely Integrable Models of Quantum Field Theory,* World Scientific Publishing, River Edge, NJ,. [A monograph on the form factor approach to two-dimensional quantum field models.]

[29] R.F. Streater and A.S. Wightman, (1964) PCT, *Spin and Statistis, and All That,* Reading, Mass., Benjamin/Cummings Publ. Co.,. [A classic account of axiomatic QFT.]

[30] F. Strocchi, (1993) *Selected Topics on the General Properties of Quantum Field Theory*, World Scientific, Singapore,. [An account of a variety of results in mathematical QFT.]

[31] S.J. Summers, (2011) Yet more ado about nothing: The remarkable relativistic vacuum state, in: *Deep Beauty*, ed. by H. Halvorson, Cambridge University Press, Cambridge, pp. 317–341. [An overview of rigorous results about the relativistic vacuum state in QFT; it includes recent developments of the important insight of Bisognano and Wichmann mentioned in Section 6.]

[32] S. Waldmann, (2007) *Poisson–Geometrie und Deformationsquantisierung*, Springer-Verlag, Berlin, Heidelberg, New York. [An account of deformation quantization.]

Biographical Sketch

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