LINEAR ELASTODYNAMICS AND WAVES

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Contents

Summary

We provide a simple introduction to wave propagation in the framework of linear elastodynamics. We discuss bulk waves in isotropic and anisotropic linear elastic materials and we survey several families of surface and interface waves. We conclude by suggesting a list of books for a more detailed study of the topic.

1. Introduction

In elastostatics we study the equilibria of elastic solids; when its equilibrium is disturbed, a solid is set into *motion*, which constitutes the subject of elastodynamics.

Early efforts in the study of elastodynamics were mainly aimed at modeling seismic wave propagation. With the advent of electronics, many applications have been found in the industrial world. These include the manufacturing of high frequency acoustic wave filters and transducers (used in everyday electronic devices such as global positioning systems, cell phones, miniature motors, detectors, sensors, etc.), the health monitoring of elastic structures (non-destructive evaluation in the automotive or aeronautic industry), the acoustic determination of elastic properties of solids (physics, medicine, engineering, etc.), ultrasonic imaging techniques (medicine, oil prospection, etc.), and so on.

The local excitation of a body is not instantaneously detected at a distance away from the source of excitation. It takes time for a disturbance to propagate from one point to another, which is why elastodynamics relies heavily on the study of *waves*. Everyone is familiar with the notion of wave, but the broad use of this term makes it difficult to produce a precise definition. For this reason we shall consider the notion of wave as primitive.

A fundamental mathematical representation of a wave is

$$u(x,t) = f(x - vt) \tag{1}$$

where f is a function of the variable $\xi = x - vt$ and v is a nonzero constant. Waves represented by functions of the form (1) are called *traveling waves*. For such waves the initial profile $u(x,0) = f(x)$ is translated along the $x-$axis at a speed $|v|$. For this reason traveling waves are also called *waves of permanent profile* or *progressive plane waves*.

Traveling waves are a most important class of functions because the general solution of the classical one-dimensional wave equation: $u_{tt} = v^2 u_{xx}$ is the sum of two of such waves, one, $F(x - vt)$, moving right with speed v, and the other, $G(x + vt)$ moving left, also with speed v:

$$u(x,t) = F(x - vt) + G(x + vt), \tag{2}$$

where F and G are arbitrary functions. Therefore the solution to any initial value problem in the entire real line $-\infty < x < \infty$ of the wave equation can be written in terms of such two traveling waves via the well known d'Alembert form.

When the functions F and G in (2) are sinusoidal we speak of *plane harmonic waves*. This is the case if

$$f = A \cos k(x - vt), \tag{3}$$

where A, k, v are constant scalars. The motion (3) describes a wave propagating with *amplitude* A, *phase speed* v, *wavenumber* k, wavelength $2\pi/k$, angular *frequency* $\omega = kv$, and temporal period $2\pi/\omega$. For mathematical convenience, the wave (2) can be represented as

$$f = \{A \exp \mathrm{i}k(x - vt)\}^{+}, \tag{4}$$

where $\{\bullet\}^{+}$ denotes the real part of the complex quantity.

In a three-dimensional setting, a plane harmonic wave propagating in the direction of the unit vector $\mathbf{n}$ is described by the mechanical displacement vector,

$$\mathbf{u} = \{\mathbf{A} \exp \mathrm{i}k(\mathbf{n} \cdot \mathbf{x} - vt)\}^{+}, \tag{5}$$

where $\mathbf{A}$ is the amplitude vector, possibly complex. If $\mathbf{A}$ is the multiple of a real vector: $\mathbf{A} = \alpha \mathbf{a}$ say, where α is a scalar and $\mathbf{a}$ a real unit vector, then the wave is *linearly polarized*; in particular when $\mathbf{a} \times \mathbf{n} = 0$, the wave is a linearly polarized *longitudinal wave*, and when $\mathbf{a} \cdot \mathbf{n} = 0$, the wave is a linearly polarized *transverse wave*. Otherwise (when $\mathbf{A} \neq \alpha \mathbf{a}$) the wave is *elliptically polarized*; in particular when $\mathbf{A}.\mathbf{n} = 0$, the wave is an elliptically polarized transverse wave.

A three-dimensional setting allows for the description of more complex wave phenomena; for example it is possible to investigate the following interesting generalization of the solutions in (5),

$$\mathbf{u} = \{g(\mathbf{m} \cdot \mathbf{x}) \mathbf{A} \exp \mathrm{i}k(\mathbf{n} \cdot \mathbf{x} - vt)\}^{+}, \tag{6}$$

where $\mathbf{m}$ is another unit vector and g is the amplitude function. The planes $\mathbf{n} \cdot \mathbf{x} = \text{constant}$ are the *planes of constant phase*, and the planes $\mathbf{m} \cdot \mathbf{x} = \text{constant}$ are the *planes of constant amplitude*. When $\mathbf{n}$ and $\mathbf{m}$ are parallel, the waves are said to be *homogeneous*; this class includes (5) as a special case, but also waves for which the amplitude varies in the direction of propagation, such as attenuated homogeneous waves, see Figures 1. When $\mathbf{n} \times \mathbf{m} \neq 0$, the waves are *inhomogeneous*; this class includes a wave which propagates harmonically in one direction while its amplitude decays in another, see Figures 2.

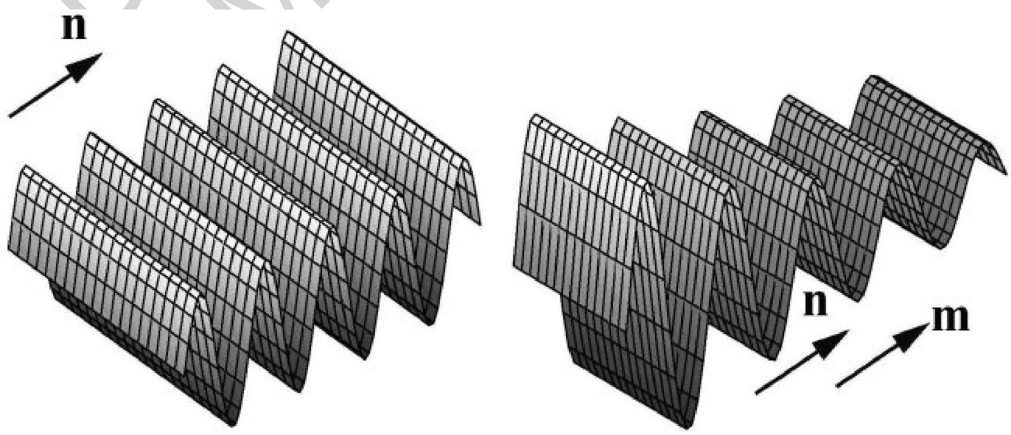

Figure 1. Homogeneous waves. On the left: a wave with constant amplitude; On the right: a wave with an attenuated amplitude.

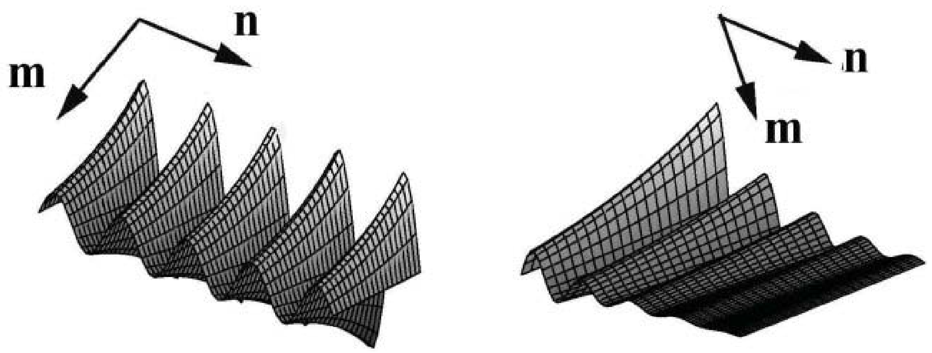

Figure 2. Inhomogeneous waves. On the left: the planes of constant phase are orthogonal to the planes of constant amplitude; On the right: they are at 45°.

In the case of homogeneous waves (5), note that instead of the *wave vector/speed* couple $(\mathbf{k}, v)$ where $\mathbf{k} = k\mathbf{n}$, the *slowness vector/frequency* couple $(\mathbf{s}, \omega)$ where $\mathbf{s} = v^{-1}\mathbf{n}$, can be used equivalently, giving the representation:

$$\mathbf{u} = \{\mathbf{A} \exp i\omega(\mathbf{s} \cdot \mathbf{x} - t)\}^{+}. \tag{7}$$

Similarly for inhomogeneous plane waves, a complex slowness vector can be introduced: $\mathbf{S} = \mathbf{S}^{+} + i\mathbf{S}^{-}$, giving the waves:

$$\mathbf{u} = \{\mathbf{A} \exp i\omega(\mathbf{S} \cdot \mathbf{x} - t)\}^{+}, \tag{8}$$

as a sub-case of (6). Here ω is the real angular frequency, the planes of constant phase are $\mathbf{S}^{+} \cdot \mathbf{x} =$constant and the planes of constant amplitude are $\mathbf{S}^{-} \cdot \mathbf{x} =$constant. The phase speed is $v = 1/\left|\mathbf{S}^{+}\right|$ and the attenuation factor is $\left|\mathbf{S}^{-}\right|$.

As a train of waves propagates, it carries energy. It can be shown that for time-harmonic waves, the ratio of the mean energy flux to the mean energy density, $\mathbf{v}_{\mathrm{g}}$ say, is computed as

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}}, \qquad \left[\mathbf{v}_g\right]_i = \frac{\partial \omega}{\partial k_i}. \tag{9}$$

This vector is called the *group velocity*.

When we consider the wave equation in a semi-infinite domain or in a finite domain, the resolution can become quite complex, because we now have to satisfy not only initial conditions but also *boundary conditions*. In this chapter we study some wave solutions to the equations of elastodynamics in the case of an infinite medium, and then for some simple boundary conditions, with a view to demonstrate the usefulness and versatility of homogeneous and inhomogeneous plane waves.

2. Bulk Waves

We start by considering the propagation of waves in an infinite elastic medium. They are often called *bulk waves,* because they travel within the bulk of a solid with dimensions which are large compared to the wavelength, so that boundary effects can be ignored (think for instance of the waves triggered deep inside the Earth crust by a seismic event.)

Referred to a rectangular Cartesian coordinate system $(Ox_1x_2x_3)$, say, the particle displacement components are denoted (u_1,u_2,u_3), and the strains are given by

$$2\varepsilon_{ij} \equiv u_{i,j} + u_{j,i}, \tag{10}$$

where the comma denotes partial differentiation with respect to the Cartesian coordinates x_j.

The constitutive equations for a general anisotropic homogeneous elastic material are

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \tag{11}$$

where the elastic stiffness parameters c_{ijkl} are constants, with the symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{12}$$

Because of these symmetries there are at most twenty-one independent elastic constants. The equations of motion, in the absence of body forces, read

$$\text{div}\,\boldsymbol{\sigma} = \rho\partial^2\mathbf{u}/\partial t^2, \tag{13}$$

where ρ is the mass density.
Inserting the constitutive equation (11) into the equations of motions (13), we obtain

$$c_{ijkl}u_{k,lj} = \rho\partial^2 u_i/\partial t^2. \tag{14}$$

For *isotropic* elastic materials, 12 elastic stiffness parameters are zero, and the remaining 9 are given in terms of 2 independent material constants: λ and μ, the so-called *Lamé coefficients*. In that case the c_{ijkl} can be written as

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{15}$$

Then the constitutive equations (11) reduce to

$$\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{16}$$

and the equations of motions (14) read

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} = \rho \partial^2 u_i / \partial t^2 . \tag{17}$$

Writing down the governing equations for *anisotropic* elastic materials is a more complex operation than what we have just done for isotropic materials.

For example let us consider *transversely isotropic materials*. They have 12 elastic stiffness parameters which are zero, whilst the remaining 9 can be expressed in terms of 5 independent material constants. When the axis of symmetry is along x_3, the constitutive equations (11) read

$$\begin{aligned}
\sigma_{11} &= d_{11} u_{1,1} + d_{12} u_{2,2} + d_{33} u_{3,3}, \\
\sigma_{22} &= d_{12} u_{1,1} + d_{11} u_{2,2} + d_{13} u_{3,3}, \\
\sigma_{33} &= d_{13} (u_{1,1} + u_{2,2}) + d_{33} u_{3,3}, \\
\sigma_{13} &= d_{44} (u_{1,3} + u_{3,1}), \\
\sigma_{23} &= d_{44} (u_{2,3} + u_{3,2}), \\
\sigma_{12} &= \tfrac{1}{2} (d_{11} - d_{12})(u_{1,2} + u_{2,1}).
\end{aligned} \tag{18}$$

where d_{11}, d_{12}, d_{13}, d_{33}, d_{44} are independent material parameters. The equations of motion (13) now read

$$\begin{aligned}
& d_{11} u_{1,11} + \tfrac{1}{2} (d_{11} - d_{12}) u_{1,22} + d_{44} u_{1,33} \\
& \qquad + \tfrac{1}{2} (d_{11} + d_{12}) u_{2,12} + (d_{13} + d_{44}) u_{3,13} = \rho \partial^2 u_1 / \partial t^2, \\
& d_{11} u_{2,11} + \tfrac{1}{2} (d_{11} + d_{12}) u_{1,12} \\
& \qquad + d_{44} u_{2,33} + \tfrac{1}{2} (d_{11} - d_{12}) u_{2,11} + (d_{13} + d_{44}) u_{3,23} = \rho \partial^2 u_2 / \partial t^2, \\
& (d_{13} + d_{44})(u_{1,13} + u_{2,23}) + d_{44} (u_{3,11} + u_{3,22}) + d_{33} u_{3,33} = \rho \partial^2 u_3 / \partial t^2 .
\end{aligned} \tag{19}$$

As another example, consider *cubic materials*. They also have 12 elastic stiffness parameters which are zero, whilst the remaining 9 can be expressed in terms of only 3 material constants. Here the constitutive equations (11) read

$$\begin{aligned}
\sigma_{11} &= d_{11} u_{1,1} + d_{12} (u_{2,2} + u_{3,3}), \\
\sigma_{22} &= d_{11} u_{2,2} + d_{12} (u_{1,1} + u_{3,3}), \\
\sigma_{33} &= d_{11} (u_{3,3} + d_{12} (u_{1,1} + u_{2,2}), \\
\sigma_{13} &= d_{44} (u_{1,3} + u_{3,1}), \\
\sigma_{23} &= d_{44} (u_{2,3} + u_{3,2}), \\
\sigma_{12} &= d_{44} (u_{1,2} + u_{2,1}),
\end{aligned} \tag{20}$$

where d_{11}, d_{12}, and d_{44} are independent material parameters. The corresponding equations of motion read

$$\begin{aligned}
&d_{11}u_{1,11} + d_{44}(u_{1,22} + u_{1,33}) + (d_{12} + d_{44})(u_{2,12} + u_{3,13}) = \rho\partial^2 u_1/\partial t^2, \\
&d_{11}u_{2,22} + d_{44}(u_{2,11} + u_{2,33}) + (d_{12} + d_{44})(u_{1,12} + u_{3,23}) = \rho\partial^2 u_2/\partial t^2, \\
&d_{11}u_{3,33} + d_{44}(u_{3,11} + u_{3,22}) + (d_{12} + d_{44})(u_{1,13} + u_{2,23}) = \rho\partial^2 u_3/\partial t^2.
\end{aligned} \tag{21}$$

2.1. Homogeneous Waves in Isotropic Solids

Consider the propagation of homogeneous plane waves of constant amplitude in a homogeneous isotropic elastic material. Therefore, search for solutions in the form (5) to the equations of motion (17).
Introducing (5) into (17), gives the *propagation condition*

$$\mathbf{Q}(\mathbf{n})\mathbf{A} = \rho v^2 \mathbf{A}, \tag{22}$$

where the *acoustical tensor* $\mathbf{Q}(\mathbf{n})$ is given by

$$\mathbf{Q}(\mathbf{n}) = (\lambda + \mu)\mathbf{n} \otimes \mathbf{n} + \mu \mathbf{I}, \tag{23}$$

Introducing any two unit vectors $\mathbf{p}$ and $\mathbf{q}$ forming an orthonormal triad with $\mathbf{n}$ and using the decomposition $\mathbf{I} = \mathbf{n} \otimes \mathbf{n} + \mathbf{m} \otimes \mathbf{m} + \mathbf{p} \otimes \mathbf{p}$, we write

$$\mathbf{Q}(\mathbf{n}) = (\lambda + 2\mu)\mathbf{n} \otimes \mathbf{n} + \mu(\mathbf{m} \otimes \mathbf{m} + \mathbf{p} \otimes \mathbf{p}), \tag{24}$$

Non-trivial solutions to the algebraic eigenvalue problem (22) exist only if the following *secular equation*

$$\det\left(\mathbf{Q}(\mathbf{n}) - \rho v^2 \mathbf{I}\right) = 0, \tag{25}$$

is satisfied. Here it factorizes into

$$(\lambda + 2\mu - \rho v^2)(\mu - \rho v^2)^2 = 0, \tag{26}$$

giving a simple eigenvalue and a double eigenvalue. Observing (24), we find that $\mathbf{A} = \mathbf{n}$ is an eigenvector of $\mathbf{Q}(\mathbf{n})$, associated with the simple eigenvalue

$$\rho v_{\mathrm{L}}^2 = \lambda + 2\mu. \tag{27}$$

It corresponds to a linearly polarized homogeneous longitudinal bulk wave, traveling with speed v_{L}. We also find that $\mathbf{A} = \mathbf{m} + \alpha\mathbf{p}$, where α is an arbitrary scalar, is an eigenvector of $\mathbf{Q}(\mathbf{n})$ with the double eigenvalue

$$\rho v_{\mathrm{T}}^2 = \mu. \tag{28}$$

It corresponds to an elliptically polarized homogeneous transverse wave, traveling with speed v_T. Sub-cases of polarization types include linear polarization when α is real, and circular polarization when $\alpha = \pm \mathrm{i}$.

Note that the speeds do not depend on the direction of propagation $\mathbf{n}$, as expected in the case of isotropy. Note also that the wave speeds are real when

$$\lambda + 2\mu > 0, \qquad \mu > 0. \tag{29}$$

Table 1 report longitudinal and transverse bulk wave speeds as computed for several isotropic solids.

Material	ρ	λ	μ	v_{T}	v_{L}
Silica	2.2	1.6	3.1	3754	5954
Aluminum	2.7	6.4	2.5	3043	6498
Iron	7.7	11.0	7.9	3203	5900
Steel	7.8	8.6	7.9	3182	5593
Nickel	8.9	20.0	7.6	2922	6289

Table 1. Material parameters of 5 different linear isotropic elastic solids: mass density (10^3 kg/m^3) and Lamé coefficients (10^{10} N/m^2); Computed speeds of the shear and of the longitudinal homogeneous bulk waves (m/s).

2.2. Homogeneous Waves in Anisotropic Solids

Now search for solutions in the form (5) to the equations of motion in anisotropic solids (14). Then the eigenvalue problem (22) follows, now with an anisotropic acoustic tensor $\mathbf{Q}(\mathbf{n})$ with components depending on the propagation direction $\mathbf{n}$:

$$Q_{ik}(\mathbf{n}) = c_{ijkl} n_l n_j = Q_{ki}(\mathbf{n}). \tag{30}$$

In general the secular equation (25) is a cubic in v^2. It can be shown that when the roots are simple, the corresponding eigenvectors are proportional to three real vectors, mutually orthogonal. Hence in general there are three linearly polarized waves for a given propagation direction. If for some particular $\mathbf{n}$, the secular equation has a double or triple root, then a circularly-polarized wave may propagate in that direction, see the isotropic case for an example.

Consider a *transversally isotropic* solid: there the components of the acoustic tensor are found from the equations of motion (19) as

$$\begin{aligned}
&Q_{11}(\mathbf{n}) = d_{11} n_1^2 + \tfrac{1}{2}(d_{11} - d_{12}) n_2^2 + d_{44} n_3^2, Q_{12}(\mathbf{n}) = \tfrac{1}{2}(d_{11} + d_{12}) n_1 n_2, \\
&Q_{22}(\mathbf{n}) = \tfrac{1}{2}(d_{11} - d_{12}) n_1^2 + d_{11} n_2^2 + d_{44} n_3^2, Q_{23}(\mathbf{n}) = (d_{13} + d_{44}) n_2 n_3, \\
&Q_{33}(\mathbf{n}) = d_{22}(n_1^2 + n_2^2) + d_{33} n_3^2, Q_{13}(\mathbf{n}) \qquad = (d_{13} + d_{44}) n_1 n_3,
\end{aligned} \tag{31}$$

Here the secular equation (25) factorizes into the product of a term linear in ρv^2 and a term quadratic in ρv^2. The linear term gives the eigenvalue

$$\rho v_2^2 = \tfrac{1}{2}(d_{11} - d_{12})(n_1^2 + n_2^2) + d_{44} n_3^2, \tag{32}$$

and it can be checked that the associated eigenvector is $\mathbf{A} = [n_2, -n_1, 0]^{\mathrm{T}}$. It corresponds to a linearly polarized transverse wave, traveling with speed v_2. The quadratic is too long to reproduce here; in general it yields two linearly polarized waves which are neither purely longitudinal nor transverse, except in certain special circumstances, of which a few examples are presented below.

If the wave propagates along the x_1 axis, then $\mathbf{n} = [1, 0, 0]^{\mathrm{T}}$ and the secular equation factorizes into $(d_{11} - \rho v^2)(\tfrac{1}{2}(d_{11} - d_{12}) - \rho v^2)(d_{44} - \rho v^2) = 0$, giving the three eigenvalues $\rho v_1^2 = d_{11}$, $\rho v_2^2 = \tfrac{1}{2}(d_{11} - d_{12})$ (in accordance with (32)), and $\rho v_3^2 = d_{44}$. The eigenvector corresponding to ρv_1^2 is $\mathbf{A} = \mathbf{e}_1$, the unit vector along x_1, giving the wave

$$\mathbf{u} = \{\exp \mathrm{i}k(x_1 - v_1 t)\}^+ \mathbf{e}_1, \tag{33}$$

a linearly polarized longitudinal wave. The eigenvectors corresponding to ρv_2^2 and ρv_3^2 are $\mathbf{A} = \mathbf{e}_2$ and $\mathbf{A} = \mathbf{e}_3$, respectively, the unit vectors along x_2 and x_3, giving the two waves:

$$\mathbf{u} = \{\exp \mathrm{i}k(x_1 - v_2 t)\}^+ \mathbf{e}_2, \qquad \mathbf{u} = \{\exp \mathrm{i}k(x_1 - v_3 t)\}^+ \mathbf{e}_3, \tag{34}$$

two linearly polarized transverse waves.

If the wave propagates along the x_3 axis, then $\mathbf{n} = [0, 0, 1]^{\mathrm{T}}$, and the secular equation factorizes into $(d_{33} - \rho v^2)(d_{44} - \rho v^2)^2 = 0$ giving a simple eigenvalue $\rho v_1^2 = d_{33}$ and a double eigenvalue $\rho v_2^2 = d_{44}$. The corresponding solutions are a linearly polarized longitudinal wave:

$$\mathbf{u} = \{\exp \mathrm{i}k(x_3 - v_1 t)\}^+ \mathbf{e}_3, \tag{35}$$

and an elliptically polarized transverse wave

$$\mathbf{u} = \{\exp \mathrm{i}k(x_3 - v_2 t)\}^+ (\mathbf{e}_1 + \alpha \mathbf{e}_2), \tag{36}$$

where α is an arbitrary scalar. Note that a circularly polarized wave is possible for $\alpha = \pm \mathrm{i}$, as expected when the secular equation has a double root.

Directions along which circularly polarized waves exist are called the *acoustic axes*. To determine whether there are acoustic axes in a given anisotropic solid is equivalent to finding whether the secular equation admits a double root. In the present case, this could happen (a) if the determinant of the quadratic term in the secular equation is zero, or (b) if the eigenvalue (32) is also a root of the quadratic term. It can be shown that (a) is never possible, whereas (b) is always possible.

Consider a *cubic* solid: there the acoustic tensor is found from the equations of motion (21) as

$$\mathbf{Q}(\mathbf{n}) = \begin{bmatrix} (d_{11} - d_{44})n_1^2 + d_{44} & (d_{12}+d_{44})n_1 n_2 & (d_{12}+d_{44})n_1 n_3 \\ (d_{12}+d_{44})n_1 n_2 & (d_{11}-d_{44})n_2^2+d_{44} & (d_{12}+d_{44})n_2 n_3 \\ (d_{12}+d_{44})n_1 n_3 & (d_{12}+d_{44})n_2 n_3 & (d_{11}-d_{44})n_3^2+d_{44} \end{bmatrix}. \tag{37}$$

Here all three axes x_1, x_2, and x_3 are equivalent. For propagation along x_1 for instance, $\mathbf{n} = [1,0,0]^{\mathrm{T}}$ and the acoustical tensor is diagonal. There is one linearly polarized longitudinal wave traveling at speed $\sqrt{d_{11}/\rho}$, and an elliptically (and thus possibly, circularly) polarized transverse wave traveling at speed $\sqrt{d_{44}/\rho}$. Similarly when $\mathbf{n} = [0,1,0]^{\mathrm{T}}$ and when $\mathbf{n} = [0,0,1]^{\mathrm{T}}$. In each case, $\rho v_2^2 = d_{44}$ is a double eigenvalue, showing that the symmetry axes are acoustic axes.

Now take $\mathbf{n} = [\pm 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^{\mathrm{T}}$, or $\mathbf{n} = [1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3}]^{\mathrm{T}}$, or $\mathbf{n} = [1/\sqrt{3}, 1/\sqrt{3}, \pm 1/\sqrt{3}]^{\mathrm{T}}$. These directions are acoustic axes, because then

$$\mathbf{Q}(\mathbf{n}) = \tfrac{1}{3}(d_{11} - d_{12} + d_{44})\mathbf{I} + (d_{12} + d_{44})\mathbf{n} \otimes \mathbf{n}, \tag{38}$$

clearly showing the existence of a linearly polarized longitudinal wave (any $\mathbf{A}$ such that $\mathbf{A} \times \mathbf{n} = \mathbf{0}$) and of an elliptically polarized transverse wave (any $\mathbf{A}$ such that $\mathbf{A} \cdot \mathbf{n} = 0$), which can be circularly polarized. They travel with speeds v_1 and v_2, respectively, given by

$$\rho v_1^2 = \tfrac{1}{3}(d_{11} + 2d_{12} + 4d_{44}), \qquad \rho v_2^2 = \tfrac{1}{3}(d_{11} - d_{12} + d_{44}), \tag{39}$$

the first eigenvalue being simple and the second, double.

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Biographical Sketches

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