MECHANICS OF TIME DEPENDENT MATERIALS BASED ON LINEAR VISCOELASTICITY

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Summary

The fundamentals of time-dependent mechanics of isotropic, homogeneous and linearly viscoelastic body under isothermal condition are presented. Starting from the time-dependent response of the simplest two-element models with their stress-strain relations, explanation on generalized Maxwell and Voigt models under uni-axial stress and strain state are described. Since the constitutive equations in higher order differential form, are inconvenient in practical use; we introduce them into the plane of Laplace transform, paying attention to zero-condition. Moreover, the constitutive equations are also derived by the concept of fading memory in convolution integral form. Laplace transform of the equations in convolution integral yields similar stress-strain relations with those derived from the differential equations. It is shown that the constitutive equations of linearly viscoelastic body can be possibly discussed in the frame work of pseudo-elastic treatment, standing on the correspondence rule. In addition, dynamic response under sinusoidal input and the retardation of phase angle of response are discussed. The classical technique for the measurement of material functions, for example relaxation modulus, is shown step by step. In this section, the time-temperature shift technique is also introduced to discuss very wide range of time over more than ten digits. After measuring characteristic functions, the authors introduce the collocation technique with Prony series proposed by Schapery, to approximate the functions. Examples of the master curve of relaxation modulus and creep compliance are shown with theresults of approximations. The expansion of one-dimensional viscoelastic analyses to those in multi-axial stress and strain state is also presented in the framework of the correspondence rule. All equations necessary in multi-axial viscoelastic stress-strain analysis, i.e., the equilibrium equation of stress, the compatibility equation of strain, the strain-displacement relation, and boundary conditions, are Laplace transformed. In the last stage, several problems involved in simultaneous measurement of two independent material functions and in the heat generation due to loss energy under dynamic response of polymers are pointed out to encourage vigorous challenge by young reserchers and engineers.

1. Introduction

To be exact, there is no ‘time-independent’ response of material in the world. The concept of time-independence is just a simplification or abbreviation of real material response. For example, the deformation response even under an instantaneous loading imposed on a real body is clearly dependent on ‘time’. However, since the treatise on time-dependent mechanical response of materials is somewhat complicated and not so familiar even in engineering community, the treatises based on time-independent response, such as elasticity and plasticity, have been dominant in designs and analyses up to date, moreover in other words such simplification has been even useful when one handles metals and hard plastic polymers in relatively short time range.

In the recent industrial world, tremendous amount of various polymers have been developed and utilized. Our world is covered with full of polymeric materials whose mechanical properties are essentially and obviously ‘time-dependent.’ Indeed, most of polymeric solids exhibit time-dependent and viscoelastic mechanical behavior in a certain range of temperature. In order to utilize these materials practically and efficiently in proper ways, it is not only effective but necessary to understand the concept of 'time-dependent' mechanical properties and behaviors of the materials, as the first step, in a frame work of 'linear viscoelasticity.'

The word 'viscoelastic' comes from the combination of elastic nature of solids and viscous one of fluids. In an elastic body, it deforms instantaneously under load and its deformation is kept constant if load does not change. An ideal elastic body does not show any time dependent properties and behaviors, so as to say that the response of such material is essentially time-independent. On the other hand, in a viscous fluid, flow (deformation) gets on with time, and resistance against deformation depends not only on the rate of deformation but on temperature. Thus, in a viscoelastic body, both load and deformation are well dependent on time and temperature. Whether we can recognize the change of deformation or load with our natural senses is dependent both on its amount and time scale in which it occurs. The authors will discuss, in this article, the time dependent mechanical response of viscoelastic body firstly to input, then the temperature dependence of them later in a brief manner with the 'time-temperature shift' technique.

2. Model Representations of Viscoelasticity

The name of 'viscoelastic' response comes from combined 'elastic' and 'viscous' characteristics of materials. The simplest method for explaining viscoelastic behaviors will be given in the investigation of response to input in 'two-element' models.

2.1. Typical Viscoelastic Behaviors

Let us now consider a simple case of thin string under uniaxial stress and strain. For instance, under a constant stress (load), a viscoelastic body in general show time-dependent incremental strain (or deformation) accompanied with instantaneous elastic strain, which is called as 'creep.' When one deforms a body instantaneously and keeps strain constant, stress responds instantaneously up to a certain level then decreases continuously with time, called as 'stress relaxation.' Figure 1 shows a schematic illustration of the relaxation of stress response $\sigma(t)$ under strain input ε_0 imposed at $t=0$ and kept constant, hereafter.

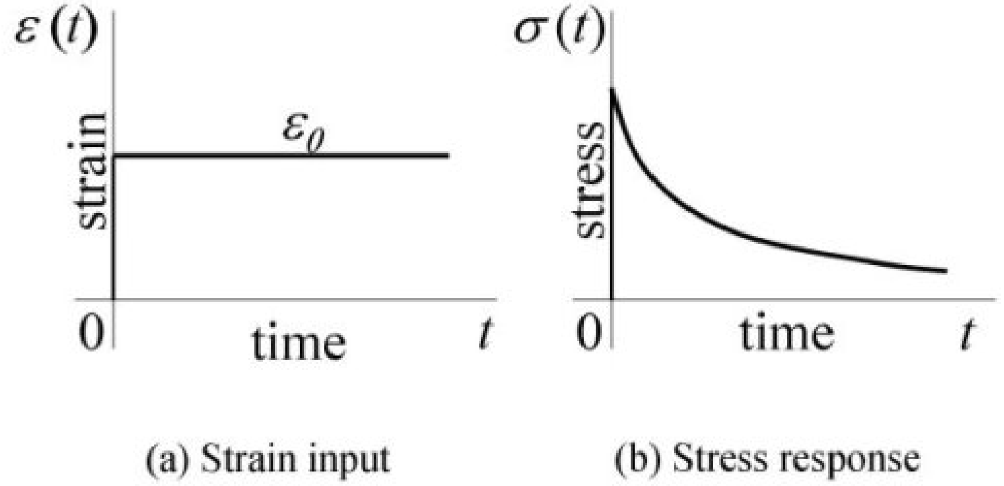

Figure 1. Schematic illustration of stress relaxation under stepwise strain input

Here we can see interesting phenomena, as an example, which will never be seen in elastic body. Let us consider an input of strain $\varepsilon(t)$ as shown in the left side of Figure 2, which can be separated as shown in the right side, standing on the superposition of stepwise strains imposed at different time. What is the response of stress $\sigma(t)$? Two curves in the left side of Figure 3 are stress responses for each input imposed at different time. Then we can superpose two response curves together as shown in the right side.

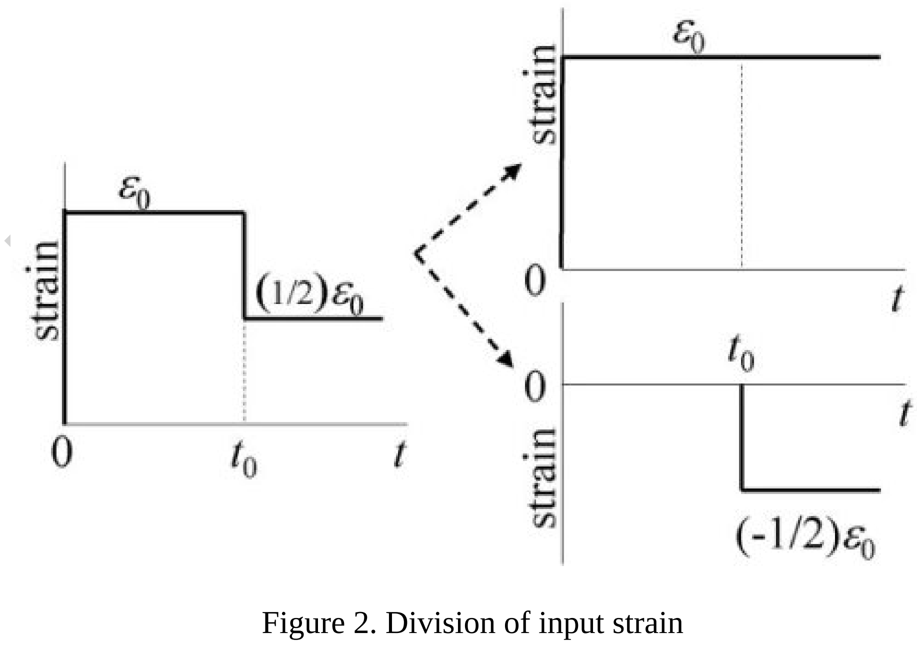

Figure 2. Division of input strain

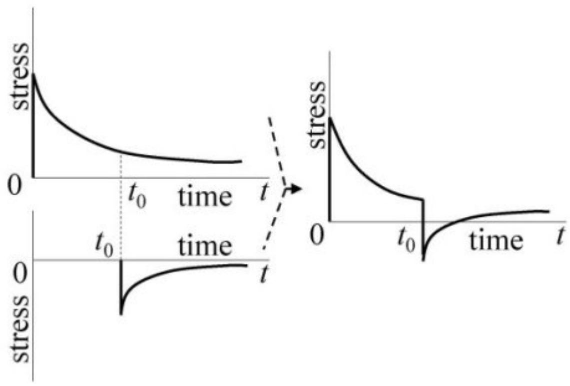

Figure 3. Superposition of responses

Here, the input strain keeps tension covering the whole time duration and does not fall into compression. But stress response sometimes drops into compression during a certain period. This is one of the typical effects of viscoelastic response. We should be, therefore, careful to the change of sign of stress due to the history of input and viscoelastic properties of material. Another typical effect is observed under dynamic or sinusoidal inputs of stress or strain. In the cases, we can observe dumping effect and/or heat generation due to loss energy which comes from stress-strain hysteresis, namely the retardation of response to input.

2.2. Constitutive Equations of Two-Element Models

Most intuitive explanation of viscoelastic responses will be given by the combination of two different types of stress-strain relation, such as Hookian elastic models, i.e. 'springs', and Newtonian viscous models, 'dashpots.' The stress-strain relations in spring and dashpot are written respectively as;

$$\sigma_{\mathrm{s}}(t) = k\varepsilon_{\mathrm{s}}(t) \quad \text{for spring,} \qquad \sigma_{\mathrm{d}}(t) = \eta \frac{d\varepsilon_{\mathrm{d}}(t)}{dt} \quad \text{for dashpot} \tag{1}$$

Here, let us combine 'spring' and 'dashpot' in series or parallel. Figure 4(a) and (b) show typical types of two-element model, i.e. Maxwell model and Voigt model.

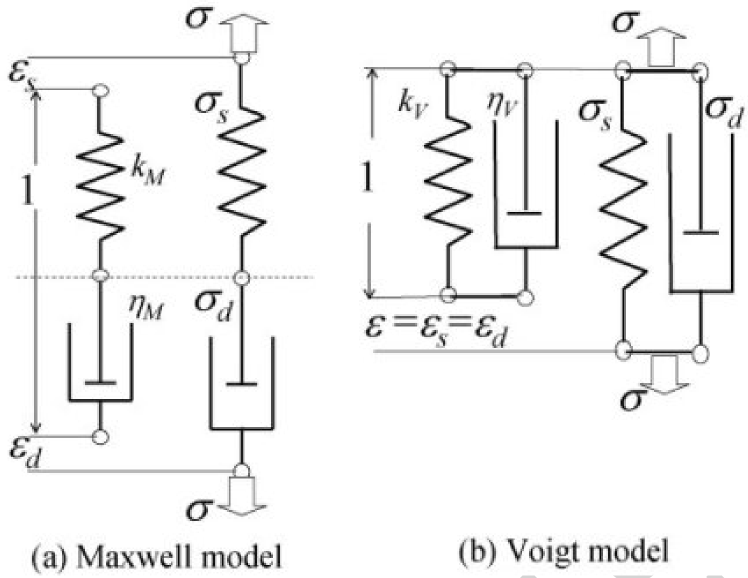

Figure 4. Two-element models

2.2.1. Maxwell Model

In this model, a spring with elastic constant k_{M} and a dashpot with viscous constant η_{M} are connected in series as shown in the figure. We consider that the model is loaded by stress $\sigma(t)$ at both ends and consequently resulted in strain $\varepsilon(t)$. Stresses in spring and dashpot are in common and total strain of those in spring and dashpot is given by the sum of them as follows,

$$\sigma_{\mathrm{s}}(t)=\sigma_{\mathrm{d}}(t)=\sigma(t), \quad \varepsilon_{\mathrm{s}}(t)+\varepsilon_{\mathrm{d}}(t)=\varepsilon(t) \tag{2}$$

Substituting Eq.(1) into the above equation, we have the stress-strain relation of the model as,

$$k_{\mathrm{M}}\frac{d\varepsilon(t)}{dt}=\frac{d\sigma(t)}{dt}+\frac{1}{\tau_{\mathrm{M}}}\sigma(t), \tag{3}$$

where $\tau_{\mathrm{M}}=\eta_{\mathrm{M}}/k_{\mathrm{M}}$, the relaxation time of Maxwell model. The meaning of τ_{M} will be shown in the later section.

2.2.2. Voigt Model

In this model as shown, a spring, k_{V}, and a dashpot, η_{V}, are arranged in parallel. Stress is divided into spring and dashpot, but strain is in common as,

$$\sigma_{\mathrm{s}}(t)+\sigma_{\mathrm{d}}(t)=\sigma(t), \quad \varepsilon_{\mathrm{s}}(t)=\varepsilon_{\mathrm{d}}(t)=\varepsilon(t) \tag{4}$$

Thus, the stress-strain relation of the model is given as,

$$\frac{d\varepsilon(t)}{dt}+\frac{1}{\tau_{\mathrm{V}}}\varepsilon(t)=\frac{1}{\eta_{\mathrm{V}}}\sigma(t) \tag{5}$$

where $\tau_{\mathrm{V}}=\eta_{\mathrm{V}}/k_{\mathrm{V}}$, the retardation time of Voigt model.

2.2.3. General Solutions; Two-element Models

Now, let us solve the differential equation Eq.(3), at first. Multiplying $e^{t/\tau_{\mathrm{M}}}$ to both sides, we have,

$$\frac{d}{dt}\left[e^{t/\tau_{\mathrm{M}}}\sigma(t)\right]=k_{\mathrm{M}}e^{t/\tau_{\mathrm{M}}}\frac{d\varepsilon(t)}{dt}$$

Changing integral variable from t to τ in the above equation and denoting the time as t at which response of the model is observed, we will have the results under the initial condition that there is no stress and strain till loading starts, as $\sigma(t)=0,\quad \varepsilon(t)=0$ at $0^{-}\geq t>-\infty$,

$$\sigma(t)=k_{\mathrm{M}}\int_{-\infty}^{t}e^{-(t-\tau)/\tau_{\mathrm{M}}}\frac{d\varepsilon(\tau)}{d\tau}d\tau=k_{\mathrm{M}}\left[\varepsilon(t)-\frac{1}{\tau_{\mathrm{M}}}\int_{-\infty}^{t}e^{-(t-\tau)/\tau_{\mathrm{M}}}\varepsilon(\tau)\,d\tau\right] \tag{6}$$

In the case of Voigt model, we can obtain the following results with similar manner, as;

$$\varepsilon(t)=\frac{1}{\eta_{\mathrm{V}}}\int_{-\infty}^{t}e^{-(t-\tau)/\tau_{\mathrm{V}}}\sigma(\tau)d\tau=\frac{1}{k_{\mathrm{V}}}\int_{-\infty}^{t}\left(1-e^{-(t-\tau)/\tau_{\mathrm{V}}}\right)\frac{d\sigma(\tau)}{d\tau}d\tau \tag{7}$$

2.3. Responses of Two-Element Models to Stepwise Inputs

Prior to carry forward our discussions on the response of the models, it will be useful to introduce the unit step function and the impulsive function defined as follows;

$$\left.\begin{aligned}U(t-t_1)&=1 \quad : t\geq t_1\\ &=0 \quad : t<t_1\end{aligned}\right\}\ \text{Unit Step Function (Heaviside Function)}$$

$$\left.\begin{aligned}\delta(t-t_1)&=\infty \quad : t=t_1\\ &=0 \quad : t\neq t_1\end{aligned}\right\}\ \text{Impulsive Function (Delta Function)}$$

For example, if we want to integrate a function of time, $f(\tau)$, from t_1 to t, using the unit function we have the results as,

$$\int_{-\infty}^{t} f(\tau)U(\tau-t_1)\,d\tau = U(t-t_1)\int_{t_1}^{t} f(\tau)d\tau \,.$$

2.3.1. Relaxation of Stress

As shown in Figure 5(a), when a stepwise strain input ε_0 at $t=0$ is given, we denote as $\varepsilon(t)=\varepsilon_0 U(t)$. In case of Maxwell model, the response is obtained from Eq.(6) as follows,

$$\sigma(t) = k_{\mathrm{M}}\varepsilon_0 U(t)\left\{1-\frac{1}{\tau_{\mathrm{M}}}\int_0^t e^{-(t-\tau)/\tau_{\mathrm{M}}}\,d\tau\right\} = k_{\mathrm{M}}\varepsilon_0 e^{-t/\tau_{\mathrm{M}}}U(t) \qquad (8)$$

The result shows an instantaneous stress response, $\sigma(0)=k_{\mathrm{M}}\varepsilon_0$, then follows up with exponential decrease with time to $\sigma(\infty)=0$. This type of stress decrease is called 'Relaxation.'

The relaxation modulus, $E_{\mathrm{r}}(t)$, i.e., stress response to unit strain, is given by,

$$E_{\mathrm{r}}(t) = \frac{\sigma(t)}{\varepsilon_0} = k_{\mathrm{M}} e^{-t/\tau_{\mathrm{M}}}U(t) \qquad (9)$$

Also, we can obtain $\sigma(\tau_{\mathrm{M}}) = k_{\mathrm{M}}\varepsilon_0/e = \sigma(0)/e$, thus the relaxation time τ_{M} represents the degree of the rate of stress relaxation. If τ_{M} is short, the relaxation occurs rapidly, and vise versa.

On the other hand, the response of Voigt model under stepwise strain input is obtained with similar calculation as,

$$\sigma(t) = \eta_{\mathrm{V}}\varepsilon_0 \frac{d}{dt}U(t) + k_{\mathrm{V}}\varepsilon_0 U(t) = \eta_{\mathrm{V}}\varepsilon_0\delta(t) + k_{\mathrm{V}}\varepsilon_0 U(t) \qquad (10)$$

The model responds like a rigid body, $\sigma(0)=\infty$, then follows up a constant stress $\sigma(t)=k_{\mathrm{V}}\varepsilon_0 U(t)$, namely no relaxation is observed in this model. Figure 5(b) shows the responses of Maxwell and Voigt models schematically.

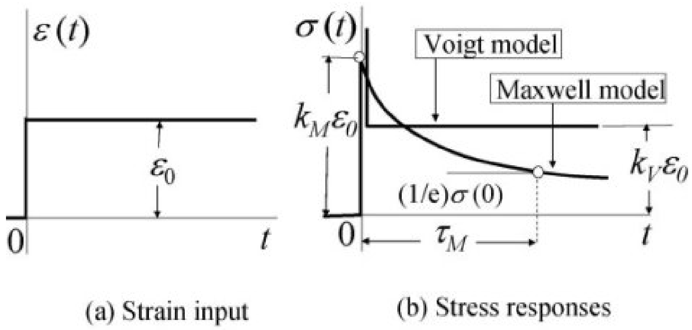

Figure 5. Stress responses of two-element models to stepwise input

2.3.2. Creep of Strain

Let us check up the creep response of the two models. Here, we will give an input stress as $\sigma(t)=\sigma_0 U(t)$. Eq.(3) gives the response of Maxwell model under the same initial condition as follows;

$$\varepsilon(t)=\varepsilon(0)+\frac{\sigma_0}{\eta_{\mathrm{M}}}U(t)\int_0^t d\tau=\varepsilon(0)+\frac{\sigma_0}{\eta_{\mathrm{M}}}tU(t), \tag{11}$$

and the creep compliance of the model is expressed as,

$$D_{\mathrm{c}}(t)=\frac{\varepsilon(t)}{\sigma_0}=\left\{\frac{1}{k_{\mathrm{M}}}+\frac{1}{\eta_{\mathrm{M}}}t\right\}U(t) \tag{12}$$

The creep response of Maxwell model shows a linear increase of strain with time, namely the dashpot is elongated endlessly.

The creep response of Voigt model is obtained by solving Eq.(7) under the same initial condition. The result is written as,

$$\varepsilon(t)=\frac{\sigma_0}{\eta_{\mathrm{V}}}U(t)\int_0^t e^{-(t-\tau)/\tau_{\mathrm{V}}}d\tau=\frac{1}{k_{\mathrm{V}}}\sigma_0\left(1-e^{-t/\tau_{\mathrm{V}}}\right)U(t) \tag{13}$$

The response strain is zero, as $\varepsilon(0)=0$ at the start of stress input, but it converge with time up to a value of $\varepsilon(\infty)=\sigma_0/k_{\mathrm{V}}$. The creep compliance of the model is shown as follows,

$$D_{\mathrm{c}}(t)=\frac{1}{k_{\mathrm{V}}}\left(1-e^{-t/\tau_{\mathrm{V}}}\right)U(t). \tag{14}$$

Figure 6 shows the creep responses of Maxwell model and Voigt model together. When evaluating the value of $\varepsilon(\tau_V)$, we can say that the retardation time, τ_V, gives the degree of rate of creep response, similarly with the case of the retardation time τ_M.

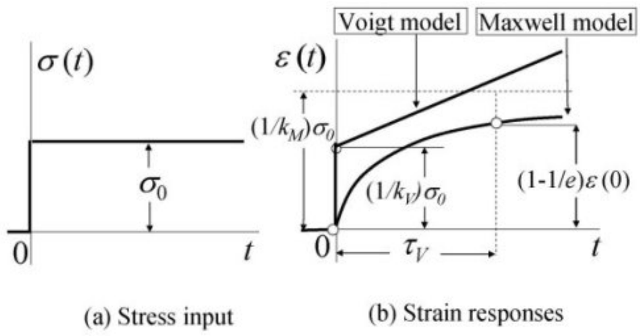

Figure 6. Strain responses of two-element models to stepwise stress input

The discussions described on two-element Maxwell and Voigt model are simple and easy to understand, since the feature of time dependent response of Maxwell or Voigt body is represented by a single relaxation time τ_M or retardation time τ_V. The two-element models are, however, simplified overly, so we have to expand discussions to more general models to describe real material properties, as described in the following section.

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Biographical Sketch

Masahisa Takashi was born on February 12, 1939, in Kagoshima Prefecture, Japan. He graduated the department of mechanical engineering, Keio University in Japan, 1963, and received Master and PhD degree of mechanical engineering from the same university in 1965 and 1969, respectively. His research interest was focused on the time-dependent mechanical behaviors and fracture of solid propellant of rocket motor in the first step, and expanded to more general viscoelastic polymers.

He had been a Lecturer, Associate Professor and Professor in the department of mechanical engineering, Aoyama Gakuin University, Japan, from 1968 to 2008. He taught strength of materials, applied elasticity and plasticity to undergraduates, and advanced course of elasticity, viscoelasticity, applied optics including photo-elasticity, photo-viscoelasticity, and fracture mechanics to graduate students. He has published more than 300 technical papers with his colleagues and students, mainly in journals and/or transactions of JSME, SEM and others.

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Akihiro Misawa: He was born on February 28, 1945, in Nagano Prefecture, Japan. He graduated the department of mechanical engineering, Keio University in Japan, 1968, and received Master and PhD degree of mechanical engineering from the same university in 1970 and 1981, respectively. His major research interests are 1) the threshold and succession conditions of crack existing in viscoelastic materials, namely how crack threshold depends on loading history and what makes succeeding growth of crack particularly in long time range, 2) the relation of crack threshold and temperature condition at which initial crack introduced.

He has been an Associate professor and Professor in the department of mechanical engineering, Kanagawa Institute of Technology, Japan, from 1981. He is teaching fundamentals of mathematics and applied elasticity to undergraduate students, and viscoelasticity with experimental techniques to graduate students. He has published more than 70 technical papers.

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