DIFFERENTIAL EQUATION MODELS

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Contents

Summary

Passive, optimal (program controlled) and regulated (feedback controlled) systems are considered. Ordinary differential equations without control and disturbance functions, with control but without disturbance function, with control and with disturbance functions represent models of considered systems.

1. Introduction

Statistic models reflect the systems, being in state of rest or equilibrium; static is symbol of invariability in time. Dynamics is state of motion and dynamic models reflect evolutions in time. This paper focuses on dynamic system models.

2. System indicators

Any investigation of system begins with problem statement, defining the set of system characteristics, or features, or indicators interesting for investigator. These are concluding or resulting indicators of investigation and in case of dynamic system they can be represented in dynamics, i.e. as depending on time.

Resulting indicators of investigation either coincide with inner indicators of system or directly connected with them. The set of inner indicators is relatively stable and as a rule isn't changed for different researches, latter varying in resulting indicators is endogenous.

At last, the system behavior depends on external influences, which are sometimes called exogenous.

So, external influences, inner indicators of system and resulting indicators of investigation are three sets characterizing the system behavior. Further, following mathematical terminology they will be called as variables: inner, external and resulting. The problem of dynamic system investigation is to define the behavior of inner variables and through them the resulting variables depending on behavior of external influences (*see Fundamentals of Mathematical Modeling for Complex Systems*).

An example of economics will make the introduced terminology more understandable. An economic element (a factory, a group of factories, a branch of economics) produces a set of products. Each produced product is an inner one in respect to the system–the economic element in this case. A part of inner product is consumed for productive needs in the element, the rest is output product. Raw materials and fund forming products, which are necessary for production and construction, are supplied from outside or the element.

The volumes of inner products in the example given above are inner variables, the volumes of output products are resulting variables, volumes of input raw materials and fund forming products are external variables or external influences.

The second example is taken from school physics: that is the system, consisting of the plane Earth and a thrown stone in this case the distance and altitude of stone flight depending on time are taken as inner variables. The altitude, distance, angle to horizon or any function of altitude and distance can be resulting variables too. In this example there are no external variables the system of Earth and stone doesn't experience external influences in case only the stone is taken as a system it will experience the gravity force as external influence.

The third example deals with rocket over the plane Earth. Here are the same inner and resulting variables, but in contrast to previous example there is an external influence in the form of rocket thrust. It is described by two external variables projection force vector on vertical coordinate and projection on horizontal coordinate.

In the further considered specific examples the variables will be designated by letters, which are usual and stable for these examples. Besides that the abstract and general cases will be considered. In these cases the inner variables are designated by symbol x_i. Index i marks number or name of the variable; it belongs to their set of names or numbers: $i \in I$. In considered here dynamic systems the variables depend on time. Sometimes this dependence will be implied, but not marked. When the fact of dynamics is necessary to underline, it will be done in the following form: $x_i(t)$, t – current time, t_0 – time of process beginning, T – its end, $t_0 \le t \le T$.

External influences may belong either to controls or to disturbances. Control influences are under command of system designer, chosen in such a way, that the system be optional. Disturbances either actively put obstacles in the realization of system goals, or indefinite for system designer. Symbol $u_k(t)$ will be appointed to controls and symbol $v_l(t)$ to disturbances.

3. Types of dynamic systems

In this article three types of dynamic systems are considered: passive, optimal program controlled and regulated feedback controlled. A passive system is not subjected either to control or counter disturbance effects. Control is present in an optimal system. Regulated system has both kinds of influences.

Dynamic systems are represented by various forms of mathematical models: ordinary differential equations, partial differential equations, intego-differential equations, difference equations. Presence of t – time argument is the important feature of these models. In this article we will discuss only those models which are described by ordinary differential equations: they are widespread and their theory is well developed.

Passive systems do not contain control or disturbance variables and are described by ordinary differential or difference equations:

$$\dot{x}_i = f_i(x_{i'}, t);\ i, i' \in I;\ x_i(t_0) = x_i^0. \tag{1}$$

In the above, i, i' – indexes of inner variables which go through the same numbers or names of set I, in the left side of the equation variable x is written with index i, and in the right side – with index i', it underlines the fact, that the function depends on all or many variables $x_{i'}$, not only x_i. The numbers x_i^0 are initial values of variables $x_i(t)$.

In contrast to passive systems optimal systems are subjected to control influences and this fact is marked by presence of control variable in the right side of differential equation:

$$\dot{x}_i = f_i(x_{i'}, u_k, t),\ i, i' \in I,\ k \in K. \tag{2}$$

Index of control variable k varies in the range of K set. In (2) the starting conditions are not written, because it is reasonable during the first acquaintance with these three types of systems to focus on their differences but not on their complete description. By the way, eq. (1) does not exhaust all the forms of mathematical models of passive systems.

Regulated systems not only have controls $u_k(t)$, but are subjected to disturbances $v_l(t)$:

$$\dot{x}_i = f_i\left(x_{i'}, u_k, v_l, t\right), \ i, i' \in I, \ k \in K, \ l \in L. \tag{3}$$

This equation is not a complete form of regulated system models; three dots at the end of the line (3) testify to this.

If we leave outside the resulting variables, and all the necessary additions which concretize the problems, it will result into (1), (2), (3), which demonstrate in relief the likenesses and differences in models of systems. Further in this article, the main results, accumulated by mathematics theory of optimality and theory of control in application to three types of dynamic systems will be set forth.

4. Examples from mechanics

4.1. Thrown stone. The first one is taken from elementary physics and related to thrown stone over the plane Earth. By letter x we will denote the distance along horizontal axis; y – altitude along vertical axis; u, v – velocity along one and another axes, g – gravity acceleration; the initial position of the stone is defined at the beginning of coordinates: $x^0 = y^0 = 0$; initial velocity vector has projections u^0, v^0 on coordinate axes. The system of differential equations and initial conditions for this problem is the following:

$$\dot{x} = u, \ \dot{y} = v, \ \dot{u} = 0, \ \dot{v} = -g;$$
$$x(0) = 0, \ y(0) = 0, \ u(0) = u^0, \ v(0) = v^0, \tag{4}$$

and its solution is

$$x = u^0 t, \ y = v^0 t - g\frac{t^2}{2}, \ u = u^0, \ v = v^0 - gt. \tag{5}$$

Excluding the parameter out of the first two expressions will lead to the function $y(x)$:

$$y(x) = \frac{v^0}{u^0}x - \frac{g}{2\left(u^0\right)^2}x^2 \tag{6}$$

which is called trajectory of free fall. The term «trajectory», or «state trajectory», or «trajectory in the state space» denotes the track of dynamic system motion in coordinate space, or state space x_i.

Although passive systems are distinguished as original class of dynamic systems, the problems formulated for them are reduced to purely mathematical sphere – as a rule, to differential equations with additional conditions at the beginning of motion if initial point is non-special, by application of one or another calculation method it is possible to get the solution of the problem. Analytical results are another affair and qualitative theory of differential equation comes here into operation. The maximal development of this theory has taken place before the age of computers set in and the results can be found in old mathematical books and journals. In any case, the systematic description of passive system theory is finished here almost right away after beginning. In the same way, the problems formulated for optimal and regulated systems are considered as solved, when they are reduced to systems of differential equations with initial conditions, i.e. to some passive systems models.

4.2. Rocket flight. The second example of mechanics concerns controlled rocket flight over the plane Earth:

$$\begin{gathered}\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = \frac{P}{m} e_x, \quad \dot{v} = \frac{P}{m} e_y - g, \quad \dot{m} = -\frac{P}{V}, \\ x(0) = 0, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 0, \quad m(0) = m^0, \\ y(T) = h, \quad u(T) = u^1, \quad v(T) = 0, \quad m(T) \Rightarrow \max, \\ e_x^2 + e_y^2 = 1, \; 0 \le P \le \overline{P}. \end{gathered} \tag{7}$$

The first and second differential equations were already introduced in (4). In the third and fourth equations are added in comparison with (4) new accelerations connected with rocket thrust P; m – mass of rocket; P/m – modules of thrust acceleration; $(P/m)e_X$, $(P/m)e_Y$ – acceleration along x and y axes; e_x, e_y – projection of unit, which is directed like thrust vector. The last differential equation describes decrease of rocket mass due to gas outflow V – outflow velocity; P/V – mass flow. Now about additional conditions: start of immovable rocket occurs from the beginning of coordinates with initial mass m^0; the task of the system is at altitude h and arbitrary distance $x(T)$ to gain horizontal velocity u^1 at zero vertical velocity; flight time is not given; final rocket mass must be maximum; thrust P can be arbitrary, but not more than $\overline{P}$.

Problem statement (7) appeared after World War II, thanks to practical task concerning artificial satellite of the Earth (see *Fundamentals of Mathematic Modeling for Complex Systems*). This problem opened a new era of optimization of dynamic systems.

5. Leontieff's balance

The second practical problem, which had so decisive influence on formation and development of general dynamic system theory, has its origin in the field of economics. It is optimization problems, involving production and consumption balances. This balances were proposed by V.V.Leontieff before World War II.

The whole national economy is divided into *n* branches: electricity, coal industry, oil industry, gas industry, machinery, etc. The branch *i* is presented by its output v_i, which is measured in product volume per time unit, as well as capacity V_i, which is maximum volume of output and conditioned by basic funds of branch *i*. Output v_i is used as raw material for branch i – v_{ii}^{σ} and other branches i' – $v_{ii'}^{\sigma}$, if i is raw materials branch. Fund forming branches, machinery and construction, give their outputs for basic funds formation capital in their branch v_{ii}^{φ} and in other branches $v_{ii'}^{\varphi}$. At least branch i brings the output part P_i to final consumption. Balances of production and consumption of branch products are

$$v_i = \sum_{i'=1}^{n} \left(v_{ii'}^{\sigma} + v_{ii'}^{\varphi} \right) + P_i \ , i = 1,...,n \, . \tag{8}$$

Now in the steps to Leontieff's balances it is necessary to express raw materials and fund forming expenditures. According to Leontieff they are:

$$v_{ii'}^{\sigma}(t) = a_{ii'} v_{i'}(t), v_{ii'}^{\varphi}(t) = b_{ii'} \dot{V}_{i'} \, . \tag{9}$$

Coefficient $a_{ii'}$ is called specific raw materials expenditure and shows the quantity of raw material i per product unit i' production. Coefficient $b_{ii'}$ is called specific fund forming expenditure and shows the quantity of fund forming product i per capacity unit i' construction.

Model of developing economy is

$$v_i = \sum_{i'=1}^{n} \left(a_{ii'} v_{i'} + b_{ii'} \dot{V}_{i'} \right) + P_i \ , v_i \le V_i \, , V_i(t_0) = V_i^0 , i = 1,...n \, . \tag{10}$$

The first equation is Leontieff's balance mentioned above; the second is limitation of the production capacity the third is initial condition for capacities.

Two new variables, which are going to be introduced, make more obvious the necessity of equations (10) to be added by criteria of economic system development:

$$\dot{V}_i = u_i \, , w_i = V_i - v_i \, ; u_i \, , w_i \ge 0 \, . \tag{11}$$

The new variable u_i is capacity V_i increase rate; it can not be negative, because the process of reconstruction, conversion and aging of basic funds isn't represented in the model and in their absence the capacity can either develop ($\dot{V}_i > 0$) or be invariant ($\dot{V}_i = 0$). The second new variable w_i is unused capacity, which can be either zero, or positive; negative excess of capacity makes no sense. Taking into account these new variables the original system changes its form but not its essence:

$$\dot{V}_i = u_i \, , v_i = \sum_{i'=1}^{n} \left(a_{ii'} v_{i'} + b_{ii'} u_{i'} \right) + P_i \ , V_i = v_i + w_i \, , V_i(t_0) = V_i^0 \, . \tag{12}$$

Being presented as (12) the system of economic equations is not included into the form (2.2), which up to now has pretended to be the general form of dynamic control systems reflection.

The first equation of (12) is a differential one, the second and the third are final connections, then come initial condition and conditions of non-negatives. When time changes are set through time derivatives (not through difference, or integral, or integral-differential operators) the enumerated composition of equations is complete, but system (12) is not general, because all the equations are linear

System (12) allows the exclusion of the variables v_i. It is more convenient to carry out the procedure of exclusion if the system (12) is presented in a vector-matrix form:

$$\dot{V} = u\,,\quad v = Av + Bu + P\,,\quad V = v + w\,,\quad V(t_0) = V^0\,,\quad u, w \ge 0\,, \tag{13}$$

where $V = (V_i)_n^1$, $v = (v_i)_n^1$, $u = (u_i)_n^1$, $P = (P_i)_n^1$, $w = (w_i)_n^1$, are vectors-columns and $A = (a_{ii'})_n^n$ $B = (b_{ii'})_n^n$ are $n \times n$ matrices.

Variable can be expressed from the second equation – Leontieff'a balance $v = RBu + RP$, where $R = (E - A)^{-1}$ and E are designation of unit matrix. This expression of v is substituted in the third balance of (13):

$$\dot{V} = u,\quad V = RBu + RP + w,\quad V(t_0) = V^0,\quad u, w \ge 0\,. \tag{14}$$

System (14) has 2*n* balances and 4*n* variables: V, u, P, w. Two ways exist to determine all the 4*n* variables.

The first one is to set 2*n* variables and define the rest of variables out of the system (14). For example: all the capacity excesses are zero ($w = 0$) and final consumption P is chosen as a kind of definite $P(t)$. The following system of differential equations turns out in this example

$$V = RB\dot{V} + RP(t),\quad V(t_0) = V^0 \tag{15}$$

not resolved in respect to derivatives. System (15) is solved by transition to the new variables: $\Phi = BV$ – volumes of basic funds.

Although the previous approach is well known, it suffers from grave shortcomings in solutions: $V(t)$ derivatives $\dot{V}$ can turn out to be negative. Another shortcoming is the necessity to form the consumption function $P(t)$. Quite another matter is optimal approach: economy moves in accordance with dynamic equations (14) sticking to non-negative conditions (14) and reaches maximum (or minimum) value of some economic criterion.

However it is necessary to note that several economic indicators (not unique) play the role of economic criterion: national income, functional on vector-function of final consumption, time of achievement of final consumption desirable levels, expenditure of socially necessary labor for achievement of capacities at desirable levels and so on. In the next section there will be considered one of the criteria of the above mentioned second type and now the problem of multicriteria for dynamic systems will be discussed.

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Bibliography

Grodzovsky G.L., Ivanov Yu.N., Tokarev V.V. (1975). *Mechanics of Space Flight. Problem of Optimization* (in Russian), Moscow: Fizmatlit Publishing Company, Science Publishers. [The jointly solution of problems of choosing of optimal project parameters of space vehicle, optimal control of engine system and optimal flight trajectory is subject of this book].

Ivanov Yu.N., Tokarev V.V., Uzdemir A.P. (1994). *Mathematical Description of Economy Elements* (in Russian), Moscow: Fizmatlit Publishing Company, Science Publishers. [Standard description of economy element is introduced. Formalized conception of material and labor flows is given].

Krasnoshzekov P.S., Petrov A.A. (1983). *Principles of Model Building* (in Russian), Moscow: Moscow University Publishers. [Methodological questions of building of mathematical models of different dynamical processes are considered].

Leontief W.W. (1986). *Input-Output Economics* (in English), New York: Oxford University Press. [The method of economy structure analysis is stated].

Biographical Sketch

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Among these the following can be mentioned:

1. *Mechanics of Space Flight. Problem of Optimization* (1975, in Russian), Moscow: Fizmatlit Publishing Company, Science Publishers (together with Grodzovsky G.L., Tokarev V.V.).
2. *Mathematical Description of Economy Elements* (1994, in Russian), Moscow: Fizmatlit Publishing Company, Science Publishers (together with Tokarev V.V., Uzdemir A.P.).
3. *Theory of Information Objects and Data Base Management Systems* (1988, in Russian), Moscow: Fizmatlit Publishing Company, Science Publishers.

Theoretical Economics: Economics Doctrines, Consumption Theory Economics (1997, in Russian), Moscow: Fizmatlit Publishing