DESCRIPTION AND CLASSIFICATION IN MIMO DESIGN

D.H. Owens and **J. Hätönen**
The University of Sheffield, United Kingdom

Keywords: Systems modeling, Control Systems Design, Multivariable Systems, Optimization Methods.

Contents

Summary

This section gives an overview of the basic building blocks of linear multivariable systems modeling and controller design. As the name implies, a linear multivariable system has several input and output variables. Furthermore, there exists linear interactions in the system, i.e. a change in the value of a particular input variable can affect the values of several output variables. These interaction effects complicate considerably the analysis and especially the design of controllers for multivariable systems.

Before any control design can take place a model of the plant in question has to be established. In the multivariable case there exist several model types that can be used to mathematically describe a dynamical multivariable system. The three most common ones are the transfer-function matrix approach, the state-space approach and the polynomial matrix approach. Each of these descriptions leads to different types of design methods, including pole-allocation, optimal control, decoupling etc.

Robustness of multivariable systems is a key issue in the controller design. This is due to the fact that multivariable systems are more sensitive to modeling uncertainty than single variable systems. Consequently robust control plays an important part in modern multivariable control.

1. Introduction

In this section, the basic building blocks of linear multivariable systems modeling and controller design are pulled together to provide an overview of the issues, the techniques and the role of theory and computation in the design process. The following sections will add detail to the story outlined here so details are limited to be sufficient only to place an idea in context and, hopefully, give the reader some insight into the motivation, possibilities and limitations that are inevitably met both in theory and in practice. The treatment is not exhaustive but the reading list should provide some access to details and related techniques not discussed here. To place some current ideas in context, a historical approach is taken from time to time – firstly to set ideas in context but secondly to ensure that the way forward takes the best from the present with the best from the past.

2. Models

2.1. Dynamical Systems and Laplace Transform

The overall goal of a feedback control is to use the principle of feedback to cause the output variable of a dynamic process to follow a given reference variable accurately regardless of any external disturbances acting on the plant. The cornerstone for feedback control design is a mathematical model of the dynamical process. These mathematical models typically consist of a set differential equations and algebraic equations. In this section the attention is restricted on linear time-invariant differential equation of the form (which belongs to the wider class of Ordinary Differential Equations or ODEs)

$$\begin{aligned} &y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\ldots+a_0y(t) \\ &=b_m u^{(m)}(t)+b_{m=1}u^{(m-1)}(t)+\ldots b_0u(t) \end{aligned} \qquad (2.1)$$

where $y^{(i)}$ $(u^{(i)})$ is the i'th derivative of $y(t)$ $(u(t))$ respect to the time variable t and for causality it is assumed that $n \geq m$. Furthermore, for the existence of a unique solution the initial conditions $y(i)(0)$ for $i=1,\ldots n$-1 have to be specified.

In this equation $y(t)$ is the measured output variable, and $u(t)$ is the input variable that can be freely manipulated and the differential equation describes the relationship between these two variables. This class of dynamical models is obtained by either analytical modeling of these systems directly from the fundamental and empirical laws governing the process or by indirect model-fitting techniques based on available plant data. The model is typically referred as a single-input single-output (SISO) system in order to differentiate it from the case where the system dynamics requires a set of differential equations to describe its behavior and consequently has more than one input and output variable.

The analysis and control design using directly the set of differential equations turned out to be a very difficult task. Hence integral transformations were introduced that transform the set of differential equations into a set of algebraic equations. The most commonly used integral transformation is the Laplace transformation defined for a time function $f(t)$, $f(t)=0$ for $t<0$, by the equation

$$L\{f(t)\} = f(s) = \int_0^\infty f(t)e^{-st}dt \qquad (2.2)$$

where s is a complex variable and the convergence of the integral is guaranteed if $f(t)$ does not grow faster than an exponential rate. The original function $f(t)$ can be recovered from its Laplace transform with the Inverse Laplace transform

$$f(t) = L^{-1}\{f(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s)ds \qquad (2.3)$$

where γ is vertical contour in the complex plane chosen so that all singularities of $f(s)$ are to the left of it. Furthermore, it can be shown that the impulse response (i.e. $u(t)$ is chosen to be the Dirac unit impulse function) of the system (1) is given by the Inverse Laplace transform of the equation

$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_0} \tag{2.4}$$

when the initial conditions of the system (1) are zero. $G(s)$ is typically termed the transfer function of the dynamical system (1). The importance of the transfer function is due to the fact that based on the Convolution Theorem, the response of the dynamical system for an arbitrary Laplace transformable input signal $u(t)$ from zero initial conditions is given by

$$\begin{aligned} y(t) &= L^{-1}\left\{y(s)\right\} = L^{-1}\left\{G(s)u(s)\right\} \\ &= L^{-1}\left\{\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_0}u(s)\right\} \end{aligned} \tag{2.5}$$

Hence for example a series and parallel connections of differential equations can be analyzed by using purely algebraic manipulations under the zero initial condition assumption.

Another important property of the transfer function is that the response of the system in steady state for a sinusoidal signal $u(t)=\sin(\omega t)$ can be calculated by setting $s = j\omega$ in the transfer function. The result is a complex number $G(j\omega)$, where the absolute value $|G(j\omega)|$ determines the resulting amplification and the angle of $G(j\omega)$ is the phase-change caused by the dynamical system, i.e. the output $y(t)$ in steady state for $u(t)=\sin(\omega t)$ is given by $y(t)=|G(j\omega)|\sin(\omega t+\arg(G(j\omega))$ where $\arg(G(j\omega))=\arctan(\mathrm{Im}(G(j\omega))/\mathrm{Re}(G(j\omega)))$. This interpretation is extremely important for example in the design of analogue filters. Furthermore, in practical applications it is relatively common that there exists experimental data of the frequency response of the system (or it is at least easy to measure) and this data can be used to identify a model of the underlying dynamical system.

2.2. State-space Equations

Another possibility for describing dynamical systems is state-space equations. In this method the differential equations a dynamic system are organized as a set of first-order differential equations in the vector-valued state of the system, and the solution is visualized as a trajectory of this state vector in an *n*-dimensional state space. The benefit of the state-space approach over the ODE approach is that it can be extended easily to systems having multiple inputs and outputs (MIMO system) and it allows the introduction of geometrical concepts into dynamical systems. For linear time-invariant systems the standard form for a state-space representation is given by

$$\begin{cases} \dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(0)=\mathbf{x}_0 \\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t) \end{cases} \tag{2.6}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbf{R}^q, \mathbf{y}(t) \in \mathbb{R}^r$ and $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the manipulated input variable and $\mathbf{y}(t)$ is the measured output variable, where each variable can be vector-valued. $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are matrices of appropriate dimension. The measurement equation reflects the fact that in most applications it is impossible to measure directly the state $\mathbf{x}(t)$ (internal variables) but one has to resort to a measurement vector $\mathbf{y}(t)$ which is a linear combination of the states.

If the original system is for example an electronic circuit that contains capacitors, inductors and resistors, then the states can be chosen to be the voltages in the capacitors and currents in the inductors, and consequently the states have a clear physical meaning. Furthermore, if the original system is given in the form of a set of time-invariant and linear ODEs, an equivalent state-space representation in terms of the input-output behavior can be always constructed, but the states do not have necessarily a clear physical meaning (see *Canonical State Space Representation*).

2.3. Transfer-function Matrices

The transfer function approach can be easily extended to cover the multiple-input multiple-output (MIMO) case. In this case the dynamical system is described with a set of linear time-invariant differential equation (for simplicity it is assumed that number of outputs is equal to the number of inputs)

$$a_{11}(p)y_1(t) + a_{12}(p)y_2(t) + \ldots + a_{1n}(p)y_n(t) = b_{11}(p)u_1(t) + b_{12}(p)u_2(t) + \ldots b_{1n}(p)u_n(t)$$

$$a_{21}(p)y_1(t) + a_{22}(p)y_2(t) + \ldots + a_{2n}(p)y_n(t) = b_{21}(p)u_1(t) + b_{22}(p)u_2(t) + \ldots b_{2n}(p)u_n(t)$$
$$\vdots$$
$$a_{n1}(p)y_1(t) + a_{n2}y_2(t) + \ldots + a_{nn}(p)y_n(t) = b_{n1}(p)u_1(t) + b_{n2}(p)u_2(t) + \ldots b_{nn}(p)u_n(t) \tag{2.7}$$

where $a_{ij}(p)$ are polynomials in the differential operator $p := d/dt$ having real coefficients. Taking the Laplace transform of this equation (assuming zero initial conditions) and solving for $\mathbf{y}(s)$ where $\mathbf{y}(s)$ is now a vector-valued variable of size $n \times 1$ gives the transfer-function representation of the dynamical system

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s) \tag{2.8}$$

where $\mathbf{G}(s)$ is a $n \times n$ transfer function matrix (TFM) where each element is a ratio $b_{ij}(s)/a_{ij}(s)$ of two polynomials in s and the off-diagonal elements represent the interaction effects between the different "channels" in the system. For the transfer function matrix case equivalent results exist as for the transfer function in Section 2.1. For the state-space representation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ in (2.5) the transfer-function matrix becomes (assuming zero initial conditions)

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2.9}$$

where the dimension of the TFM is $r \times q$ and hence a dynamical system can be described equivalently in terms of input-output behavior either with the transfer function matrix description or the state-space description.

2.4. Polynomial Matrix Models

Polynomial systems theory was motivated by the fact that differential equations can be written in the form

$$\mathbf{A}(p)\mathbf{y}(t) = \mathbf{B}(p)\mathbf{u}(t) \tag{2.10}$$

where $\mathbf{A}(p)$ and $\mathbf{B}(p)$ are polynomial matrices in the differential operator $p := \frac{d}{dt}$. If the function space where $\mathbf{y}(t)$ and $\mathbf{u}(t)$ (which are vector-valued functions) are from a correctly chosen vector space, it can be shown that the resulting algebraic structure for the polynomial operators is a ring $\mathbb{C}[p]$ where $\mathbb{C}$ is the field of complex numbers. This ring structure is strong enough for both analysis and design of controllers because the ring-structure is a Principal Ideal Domain having a division algorithm, and hence for example Gaussian elimination can be used to solve for a certain variable in the multivariable case. The advantage of this approach is that all the ambiguities with zero-pole cancellations in the Laplace-transform disappear, and the analysis and design algorithms can be implemented with a computer in a straightforward manner. Furthermore, stability, controllability and observability results coincide with the theory developed for state-space representations, resulting in a more general theory (see *Controller Design using Polynomial Matrix Description*). Both controllability and observability are connected to the existence of a solution for a Diophantine equation, where the existence of the solution can be tested efficiently with computational techniques (either numerically or symbolically).

If the system is supposed to have zero initial conditions, the system model can be written either in the left or right matrix fraction description as

$$\mathbf{y}(t) = \mathbf{A}(p)^{-1}\mathbf{B}(p)\mathbf{u}(t) = \mathbf{A}_l(p)^{-1}\mathbf{B}_l(p)\mathbf{u}(t) = \mathbf{B}_r(p)\mathbf{A}_r(p)^{-1}\mathbf{u}(t) = \mathbf{G}(p)\mathbf{u}(t) \tag{2.11}$$

where the elements of $\mathbf{G}(p)$ are now from the field of fractions $\mathbb{F}(p)$.This representation of the system can be used to derive the class of all stabilizing feedback controllers. This parameterization plays an important role in modern robust control techniques, H_2 and H_∞-theory being two examples. Furthermore, the polynomial systems approach seems to have a great potential to be one of the most suitable frameworks for multi-dimensional systems, where the number of independent axis is more than one (see *Polynomial and Matrix Fraction Description*).

2.5. Differential-delay Models

In some applications it cannot be assumed that the system can be described purely with differential equations due time-delays present in the system. The delay element(s) in the system are typically caused by transportation of material or energy over long distances, and consequently delay elements are common in flow systems in chemical engineering and communication systems. A typical example of a differential-delay system is a retarded differential-delay equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{A}_2\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \tag{2.12}$$

where the derivative of the state is not only a function of the current state $\mathbf{x}(t)$ but also a function of the delayed state $\mathbf{x}(t-\tau)$. The analysis and control design for this class of dynamical system is more complicated because the state of the system is now infinite-dimensional (in other words the state of the system is the $\mathbf{x}(s)$ for $s \in [t, t-\tau]$.

2.6. A parallel Development for Discrete-time Systems

Due the invention of the modern computer the theoretical analysis of sampled-data systems started to play an important role in control theory. This stems from the fact that digital computers can only process sampled data points $f(0), f(h), f(2h), \ldots, f(kh), \ldots$ with a sampling time h from the continuous time function $f(t)$. If it can be assumed that system input vector $\mathbf{u}(t)$ is an output of a sample-hold device (zero-order hold), then the behavior of a continuous-time state-space equation (2.5) at the sampling instances can be modeled with the discrete-time state-space equation (a difference equation)

$$\begin{cases} \mathbf{x}(t+h) = \mathbf{\Phi}\mathbf{x}(t) + \mathbf{\Gamma}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{2.13}$$

for $t = 0, h, 2h, \ldots$ and the matrices in the state-space representation (2.13) are given by the formulas

$$\mathbf{\Phi} = e^{\mathbf{A}h}, \quad \mathbf{\Gamma} = \int_0^h e^{\mathbf{A}(h-t)}\mathbf{B}dt \tag{2.14}$$

Furthermore, so called *Z*-transform plays the similar role for discrete-time linear time-invariant systems as the Laplace transform for the continuous time case, where for a discrete-signal the *Z*-transform is defined with the equation

$$Z\{f(th)\} = \sum_{k=0}^{\infty} f(kh)z^{-kh} \tag{2.15}$$

which is bounded if the signal $f(kh)$ does not grow faster than an exponentially growing function. From the definition of the *Z*-transform parallel results can be established for the transfer function $G(z)$ of a dynamical system (which also the impulse response of the system). The transfer function of the discrete state-space model (2.12) is given by

$$\mathbf{G}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma} + \mathbf{D} \tag{2.16}$$

where $\mathbf{G}(z)$ is an $n \times n$ matrix where each element is a ratio of two polynomials in z^{-1}. It is often useful to identify z with $z = e^{j\omega h}$ to relate the discrete transfer function matrix with its continuous counterpart.

The polynomial approach also covers discrete-time case, where the plant model is now written in the form

$$\mathbf{A}(q^{-1})\mathbf{y}(t) = \mathbf{B}(q^{-1})\mathbf{u}(t) \tag{2.17}$$

where q^{-1} is the backward-shift operator and $\mathbf{A}(q^{-1})$ and $\mathbf{B}(q^{-1})$ are polynomial matrices where the elements of the matrices come from the polynomial ring C[q^{-1}]. As with the continuous-time case, if the system has zero initial conditions, the system model can be written in the matrix fraction description form, and similar parameterization of all stabilizing controllers can be found.

2.7. Model Reduction and Approximation

In most analysis methods, the model is assumed to be an exact representation of the plant. This is never the case so errors in prediction must be expected. In control systems design terms, feedback is capable of suppresses model errors which leaves open the possibility that a simplified model may ease the analysis whilst still achieving the designer's objectives. The use of approximate models is hence a valuable option for design. There are many ways of obtaining an approximate model from identification methods, to simple curve fitting to more sophisticated analysis of the available model. In all cases it is normal (but not always the case) that the approximate model has a smaller state dimension than the original. The area of model reduction and approximation is large. Many applications use physical approximations to obtain a simple model. There are, however, methods that are not physically-based spanning the simple models underpinning the Ziegler-Nichols tuning method to the sophisticated methods of Hankel approximation. They are often used but all must be used with care to ensure that modeling errors do not invalidate the predictions of the design exercise. Robust control is one way of avoiding problems but this requires that some measure of the modeling error is available to the designer.

-
-
-

Bibliography

Blomberg H. & Ylinen R. *Algebraic Theory for Multivariable Control* (1983). Academic Press.[The first rigorous treatment on polynomial systems theory and MIMO systems]

Edwards J. B. & Owens D. H. (1982). *Analysis and control of multipass processes*. Chichester. MacFarlane A G J (Editor). (1979). *Frequency response methods in control systems*. IEEE Press. New York. [The first text explaining a dynamics and stability theory for repetitive processes]

Grimble M. & Kucera V. (1996). *Polynomial Methods for Control Systems Design*. Springer. [A collection of introductory topics in polynomial control systems design]

Jashmidi M., dos Santos Coelho L., Krohling R. A. and Fleming P.J. (2002). *Robust Control Systems with Genetic Algorithms*. CRC Press [A typical textbook linking optimization and search to control systems design]

Kailath T. (1980). *Linear Systems* Prentice Hall [A general textbook on multivariable systems and MFDs]

Levine W.S (1996). *The Control Handbook*, CRC Press.[A comprehensive encyclopedia in control concepts, techniques and applications]

MacFarlane A G J (Editor). (1980). *Complex variable methods for linear multivariable feedback systems.* Taylor and Francis. London.[A collection of papers bringing together major contributions to the are to 1978]

Maciejowski J M. (1989). *Multivariable feedback design. Addison-Wesley*. Wokingham.[A general textbook on multivariable feedback design ideas]

Owens D H. (1973a). *Multivariable control systems design concepts in the failure analysis of nuclear reactor spatial control systems.* Proc IEE. **120**. 119-125.[A paper linking multivariable design ideas to an application]

Owens D H. (1973b). *Dyadic approximation method for multivariable control systems analysis with a nuclear reactor application*. Proc.IEE. **120**. 801-809. [A paper linking multivariable design ideas to an application]

Owens D.H. (1978b). *Feedback and multivariable systems*. Peter Peregrinus Ltd. Stevenage UK.[A general textbook on multivariable feedback design ideas]

Owens D.H. & Chotai A. (1986). *Approximate models in multivariable process control; an inverse Nyquist array and robust tuning regulator interpretation.* Proc IEE(D). **133**. 1-12.[A paper linking step response models to inverse Nyquist array methods]

Postlethwaite I & MacFarlane A G J. (1979). *A complex variable approach to the analysis and design of linear multivariable feedback systems.* Lecture Notes in Control and Information Sciences. **12.** Springer-Verlag. Berlin.[Analysis of MIMO feedback systems using Riemann surface methods]

Rogers E & Owens D H. (1992). Stability analysis for linear repetitive processes. Springer-Verlag Series on Lecture Notes in Control and Information Sciences. **175**. Berlin. [A textbook clearing issues in a branch of repetitive control]

Rosenbrock H H. (1969). *Design of multivariable feedback systems using the inverse Nyquist array.* Proceeding IEE, **116**, 1929-1936.[The original paper introducing the inverse Nyquist array to the community]

Skogestad S. & Postlethwaite (1996). *Multivariable Feedback Control – Analysis and design*. Wiley.[A textbook on robust multivariable control]

Biographical Sketches

David Owens is Professor and Head of the Department of Automatic Control and Systems Engineering and Dean of Engineering at Sheffield University. He has over 30 years involvement in academic life with experience of departmental teaching, administration and research and high level University committee service in three UK HE Institutions. His research interests have included aspects of multivariable and robust control, adaptive control, optimization theory and repetitive and iterative learning control.

Jari Hätönen works as a Research Associate at Department of Automatic Control and Systems Engineering, University of Sheffield. His research interests are Repetitive Control, Iterative Learning Control and polynomial systems theory.