PONTRYAGIN'S MAXIMUM PRINCIPLE

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Contents

Summary

The standard mathematical model of a controlled system is usually given by an ordinary differential equation with additional parameters in its right-hand side – the "controls". These are to be selected so as to ensure a pre-specified course ("goal") of the process on one hand and to optimize a given cost function ("performance index") on the other. This leads in general to variational problems with constraints.

In applied problems the constraints may often be given in the form of inequalities, while the cost functions and other items may be non-smooth. Such problems require techniques which are beyond the scope of classical calculus of variations.

The set of necessary conditions for optimality in such problems are given by Pontryagin's Maximum Principle and its modifications which generalize classical methods. The present article describes the Maximum Principle as well as the range and scheme of its application. It also indicates some possible generalizations.

1. Introduction

Pontryagin's Maximum Principle is a proposition which gives relations for solving the variational problem of optimal open-loop control. In general that is a non-classical variational problem which allows treatment of functions and constraints that are beyond those considered in classical theory, but are very natural for practical problems.

The Maximum Principle was formulated in 1956 by L.S. Pontryagin (Lev Semenovich Pontryagin (1908 – 1988) – Russian mathematician, Member of the Academy of Sciences of the USSR, distinguished world-wide for his works in topology and differential equations as well as in control theory and differential games) and further developed by himself and his associates, as well as in many other investigations.

It was motivated by new problems in automation and aerospace engineering, initiating the "mathematical theory of controlled processes". The maximum principle was and is broadly used for solving applied problems of control and other problems of dynamic optimization. It has triggered numerous generalizations and applications. The basic necessary conditions from classical Calculus of Variations follow from the Maximum Principle.

In many Western publications the Maximum Principle of Pontryagin is also referred to as "the Minimum Principle" (by changing signs in some of the upcoming relations the "maximum condition" of the sequel may be rewritten in the form of a minimum condition).

2. An Example

Suppose a car of unit mass is free to move along a horizontal track (the axis x) under the influence of a control force (a thrust) $u_1(t)$ being also subject to a frictional force $k + u_2$, so that its motion is described by the differential equation

$$\frac{d^2x}{dt^2} + (k + u_2(t))\frac{dx}{dt} = u_1(t),$$

or, taking $x_1 = x,\ x_2 = dx/dt,$ by the system

$$\dot{x}_1 = x_2,\ \ \dot{x}_2 = -kx_2 - u_2(t)x_2 + u_1(t).$$

Here $k > 0;\ 0 \le u_2(t) \le 1,$ while $u_1(t)$ is unbounded.

The forces $u_1(t)$, $u_2(t)$ are the controls which have to be selected. The control task is to move the car from starting point $x_1(0) = 0,\ x_2(0) = 0$ to target point $x_1(T) = a,\ \ x_2(T) = 0,$ ensuring a "soft" stop with velocity zero.

The controls $u_1(t)$, $u_2(t)$ have to be chosen such that the control task would be fulfilled with minimum fuel consumption, presuming the consumption is proportional to the integral

$$J = \int_0^T u_1^2(t)dt.$$

Here to move we have to consume the fuel $(u_1(t))$ while an additional regulation of the motion may be done by applying the brakes $(u_2(t))$. We presume that applying the brakes does not affect the fuel consumption. The range of force $u_2(t)$ is bounded.

Thus the problem is to find the optimal controls $u_1(t)$, $u_2(t)$ while fulfilling the task and minimizing the fuel consumption J. Here the force $u_2(t)$ does not explicitly affect the consumption J, but applying it at some time may allow to switch off the thrust $u_1(t)$ and thus help to save fuel. The given problem is a variational problem, but is not directly treated in classical theory because of the inequality constraint on $u_2(t)$ (classical problems usually deal only with equality constraints).

In order to solve non-classical variational problems one may apply the Maximum Principle.

We may now proceed with a more general formulation.

3. The Problem of Optimal Control

A typical problem of optimal open-loop control sounds as follows. Given are the vector-valued system equations

$$\frac{dx}{dt} = f(x,u), \tag{1}$$

or, in more detail,

$$\frac{dx_i}{dt} = f_i(x,u), \quad (i = 1,\ldots,n),$$

where $x \in \mathbb{R}^n$ is the n-dimensional state of the system and $u \in \mathbb{R}^m$ - the m-dimensional control. Also given are the starting point $x^{(0)}$ and the terminal point $x^{(1)}$:

$$x(0) = x^{(0)},\ x(\tau) = x^{(1)}. \tag{2}$$

Relations (2) are the boundary conditions. The range of the control is the constraint set $\mathcal{P}$

Problem I of Optimal Open-Loop Control – OOLC- is to find such a function $u(t)$ which would steer the system from starting point $x^{(0)}$ to terminal point $x(\tau) = x^{(1)}$ under the constraint

$$u(t) \in \mathcal{P}, \tag{3}$$

while minimizing an integral cost functional

$$J(x(\cdot),\ u(\cdot)) = \int_0^{\tau} f_0(x,u)dt = \min_u . \qquad (4)$$

Here $x(\cdot)$ stands for the whole function $x(t),\ t \in [0,\tau]$. The terminal point $x(\tau) = x^{(1)}$ may be substituted by a terminal target set $\mathcal{M}$ and the time τ may be either fixed or free. Note that the term "cost functional" is usually applied to problems of minimization while for maximization problems the term "performance index" is more common.

The fact that the control $u = u(t)$ is selected among functions of time t indicates that we have the problem of open-loop control. This is in contrast with problems of closed-loop or feedback control where the control is chosen as a function $u = u(t,x)$ of both time and state.

The problem in which $f_0(x,u) \equiv 1$ and the time τ is free brings us to $J = \tau = \min_u$. This is the time-optimal control problem where one has to select a control $u(t)$ which steers the system from $x^{(0)}$ to $x^{(1)}$ in minimal time.

We shall say that $u(t), x(t)$ is a pair if $u(t)$ is the control which generates the trajectory $x(t)$ of (1), with $x(0) = x^{(0)}$.

The solution to the Problem of OOLC is the pair $\{u^0(t),\ x^0(t)\}$, where $u^0(t)$ is the optimal control and $x^0(t)$ - the optimal trajectory. Clearly, the optimal control must satisfy the constraint (3) and the optimal trajectory must satisfy the boundary condition (2). The pair $\{u^0(t),\ x^0(t)\}$ must minimize the cost criterion (4). (In this chapter we always presume that our problem is to minimize J. If necessary to maximize J, we must have to minimize $-J$).

Loosely speaking, Pontryagin's Maximum Principle gives the necessary conditions for a control $u^0(t)$ to be the OOLC.

4. A More Rigorous Formulation of the Problem

This section is for a more scrupulous reader, and may be skipped. (One should realize, though, that a truly rigorous mathematical description should be even more detailed than here).

For a more rigorous mathematical setting one should first describe the type of functions $f(x,\ u), f_0(x,\ u)$ (the class of system models and cost functions) and the class of controls $u(t)$ for which the problem would make sense and would be solvable through the proposed technique.

The standard conditions for $f(x,\ u)$ are that the components $f_i(t,\ x)$ of

$f(t, x), i = 1,\dots, n,$ as well as the scalar function $f_0(t, x)$, are continuous in both variables x, u and continuously differentiable in x, u. The admissible controls $u(t)$ are assumed to be piecewise-continuous, with finite number of discontinuities. To be precise, in the forthcoming formulations they are assumed to be right-continuous with no discontinuities at the starting and final instances of time. The class of such functions, constrained by (3) and defined within the interval $[0, \tau]$, is denoted as U.

The constraint set $\mathcal{P}$ is taken to be closed and bounded. This is typical for engineering problems. The admission of closed constraint sets makes the variational problem (1)-(4) of optimal control non-classical and places it beyond the scope of classical Calculus of Variations.

In variational problems one distinguishes local minima from global minima and strong minima from weak minima. A functional $J(x(\cdot), u(\cdot))$ is said to attain a strong local minimum at $x^0(\cdot), u^0(\cdot)$ if there exists an number $\delta > 0$ such that the inequality

$$J(x^0(\cdot), u^0(\cdot)) \le J(x(\cdot), u(\cdot))$$

is true for all trajectories $x(t)$ that satisfy the boundary condition (2) and the inequality

$$\left\| x(t) - x^0(t) \right\| \le \delta, \quad \forall t \in [0, \tau],$$

being generated by controls $u(\cdot)$ that are in U and deviate from $u^0(\cdot)$ on a finite number of intervals of small total measure. (Such "variations" of controls are known as "needle-type" and were introduced by J.McShane).

Pontryagin's Maximum Principle is the statement of necessary conditions for a strong local minimum (maximum) in the problem of OOLC.

What we avoid to discuss here is whether the solution to the Problem of OOLC does exist at all in the class of controls U. This is a separate question of controllability of system (1), (3) (see Section 7, Definition I of this article) and of the properties of function $f_0(x, u)$, which have to ensure that the minimum of integral cost (4) may be attained within the class U.

The formulation of the maximum principle presumes that the solution to the Problem of OOLC does exist indeed.

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Remark. References marked with asterisk * are for advanced readers.

Biographical Sketch

Alexander B. Kurzhanski was born on 19.10.1939. He is married and has two sons. He graduated in Electrical Engng. from Technical University of Ural in 1962 and had graduate studies in mathematics at University of Ural , both at Yekatherinburg (Sverdlovsk) Russia - 1962-1965.

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