$\mathcal{H}_\infty$ OPTIMAL CONTROL

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Keywords: all sublevel solutions, complementary sensitivity function, generalized plant, $\mathcal{H}_\infty$ norm, $\mathcal{H}_\infty$ optimization, input sensitivity function, Kalman-Jacubovič-Popov formula, LQG optimization, minimum sensitivity problem, mixed sensitivity problem, model matching problem, optimal solutions, performance, Riccati equation, robust control system design, robustness, sensitivity function, spectral factorization, standard $\mathcal{H}_\infty$ problem, state space solution, sublevel solutions.

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Summary

H_∞-infinity optimization is a design method for control systems based on minimization of the H_∞-infinity norm of the closed-loop transfer matrix. Both closed-loop performance and robustness may be characterized in terms of this norm. The method therefore has considerable potential for dealing with both aspects of control system design.

To elucidate this a brief overview is given of the importance of closed-loop functions such as the sensitivity function and the complementary sensitivity function in analyzing performance and robustness. This leads to the mixed sensitivity problem, which in turn is a special case of the standard H_∞-infinity problem.

All solutions of the standard H_∞-infinity problem rely on first determining sublevel solutions. Optimal solutions are approached by a line search on the level.

The standard H_∞-infinity problem may be solved by spectral factorization. The state space solution relies on the solution of two algebraic Riccati equations. Many other variants of the H_∞-infinity problem have been studied.

The article concludes with a brief review of the importance of H_∞-infinity methods for robust control system design.

1. Introduction

$\mathcal{H}_\infty$ optimization was introduced in the control field by the American engineer and applied mathematician George Zames (1934–1997) in the late 1970s. The subject fits in the major stream of work initiated by Norbert Wiener (1894–1964) to cast design problems as mathematical optimization problems. The idea is to express the success of a tentative design in meeting the design objective as a numerical value. Then all that is needed to obtain the best design is to solve the mathematical problem of maximizing (or minimizing, as the case may be) this number with respect to the free parameters in the design.

Wiener, who profoundly affected many developments in systems and control theory, developed this idea during the Second World War for filter design, but it was soon recognized that it could be used for feedback control system design as well. After a somewhat unfortunate period in the history of control, when during the 1960s control system design was confused with trajectory optimization, the important work of Rudolph E. Kalman on what became known as LQG optimization refocused the control community's interest on feedback as its most essential feature. Kalman was an American control theorist whose work on optimal filtering and control in the late 1950s and 1960s was decisive for the development of systems and control theory.

A very powerful property of feedback control is that a properly designed feedback system with sufficiently high gain is intrinsically robust, that is, insensitive with respect to uncertainties and variations in the plant dynamics. Although Kalman was the first to recognize that LQG-optimal feedback systems have very favorable robustness properties, the formulation of the problem itself does not address robustness at all but solely involves performance expressed in terms of mean square errors. Only a thorough understanding of the robustness properties of LQG-optimal systems, the optimization criterion, makes it possible to comply with robustness design targets, be it indirectly.

When Zames formulated the first version of what became known as $\mathcal{H}_\infty$ optimization his ambition was to make robustness an intrinsic feature of the optimization problem. In his early lectures on the subject he announced his results as a design solution that would definitively eliminate the trial and error approach that characterizes other optimization approaches to control system design. It is interesting to observe that these high hopes in the end were not met even though $\mathcal{H}_\infty$ optimization developed to a powerful design method. (See also *Optimal Linear Quadratic Control (LQ), and LQ-Stochastic Control*.)

2. The Minimum Sensitivity Problem

To introduce $\mathcal{H}_\infty$ optimization consider the simple single-input–single-output feedback loop of Figure 1. If the signal v represents the disturbance that acts on the feedback loop and L is the loop gain (expressed as a Laplace transfer function) then the output signal z of the feedback system is given by

$$z = \frac{1}{1+L} v \tag{1}$$

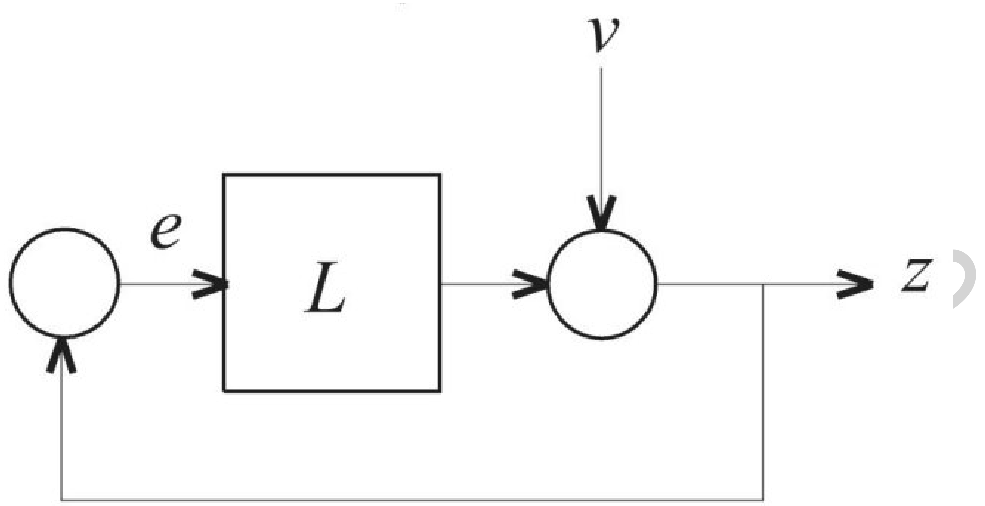

Figure 1. SISO feedback loop

This relation defines the sensitivity function

$$S = \frac{1}{1+L} \tag{2}$$

of the closed-loop system. The sizes of the disturbance v and the output signal z may be represented by their well-known 2-norms

$$\|v\|_2 = \sqrt{\int_{-\infty}^{\infty} |v(t)|^2 \, dt}, \quad \|z\|_2 = \sqrt{\int_{-\infty}^{\infty} |z(t)|^2 \, dt} \tag{3}$$

Given these signal norms, the norm of the function that maps the disturbance v to the output z may be defined as

$$\|S\| = \sup_{\|v\|} \frac{\|z\|_2}{\|v\|_2} \tag{4}$$

Obviously, whatever the disturbance v is, the norm of the output z is always bounded by

$$\|z\|_2 \le \|S\| \cdot \|v\|_2 \tag{5}$$

Thus, if we manage to make the norm $\|S\|$ as small possible then we may be sure that the norm of the output signal is as small as possible given the norm of the disturbance. This is the argument that led Zames to consider the “minimum sensitivity problem,” the first $\mathcal{H}_\infty$ optimization problem that was studied.

A simple mathematical argument based on Parseval's theorem from Fourier analysis reveals that

$$\|z\|_2^2 = \int_{\infty}^{-\infty} |\, z(t)\,|^2 \, dt = \int_{-\infty}^{\infty} |\, \hat{z}(f)\,|^2 \, df = \int_{-\infty}^{\infty} |\, S(j2\pi f)\hat{v}(f)\,|^2 \, df \qquad (6)$$

is minimal if

$$\|S\| = \sup_f |\, S(j2\pi f)\,| \qquad (7)$$

is minimal. Here $\hat{z}$ and $\hat{v}$ are the Fourier transforms of *z* and *v*, respectively.

Zames solved the problem of minimizing $\|S\|$ by interpolation theory. Simple examples soon showed that the minimum sensitivity problem as stated usually leads to unrealistic feedback systems with infinite bandwidth, often with infinite gain. Inspection of $S = 1/(1+L)$ confirms the well-known fact that because for any real-world system the loop gain *L* vanishes at high frequencies (where the precise meaning of "high" depends of course on the particular system) the sensitivity function approaches 1 for these same high frequencies. This led Zames to modify the sensitivity problem to the "minimum weighted sensitivity problem," which amounts to minimizing an expression of the form

$$\|SW\| = \sup_f |\, S(j2\pi f)W(j2\pi f)\,| \qquad (8)$$

W is a suitable weighting function. Clearly, this approach is rather unsatisfactory because the acceptability of the design crucially depends on the choice of the weighting function *W*. This led to the study of other $\mathcal{H}_\infty$ problems, in particular "the mixed sensitivity problem" discussed in Section 4.

The system norm $\|S\|$ defined by Eq. (4) is said to be *induced* by the 2-norm on the input and output signals. Because of the explicit representation Eq. (7) of this norm, which is similar to the definition of the ∞-norm of a signal, the system norm is usually referred to as the ∞-norm of the system with transfer function *S*, and correspondingly denoted as

$$\|S\|_\infty = \sup_f |\, S(j2\pi f)\,| \qquad (9)$$

The name $\mathcal{H}_\infty$ optimization problem refers to the fact that the problem may be viewed as an optimization problem over functions defined on the Hardy space $\mathcal{H}_\infty$ of stable causal frequency response functions. G.H. Hardy (1877–1947) was a famous British pure mathematician working on number theory who used to boast of his certainty that his work would never find any practical application. (See also *Closed-Loop Behavior*.)

3. Robustness and the Sensitivity Functions

In the same period when Zames developed his ideas John C. Doyle, another American control theorist, presented some ideas on robustness which were no doubt partly inspired by Zames' earlier work on the applications of the small gain theorem in network and control theory. The small gain theorem states that a sufficient condition for the closed-loop system of Figure 1 to be stable is that the norm $\|L\|$ of the loop gain is less than 1. Doyle applied this criterion to study the robust stability of the configuration of Figure 2. In this diagram the block Δ_L is called a “proportional” perturbation of the loop gain because it modifies the loop gain from L to $(1+\Delta_L)L$.

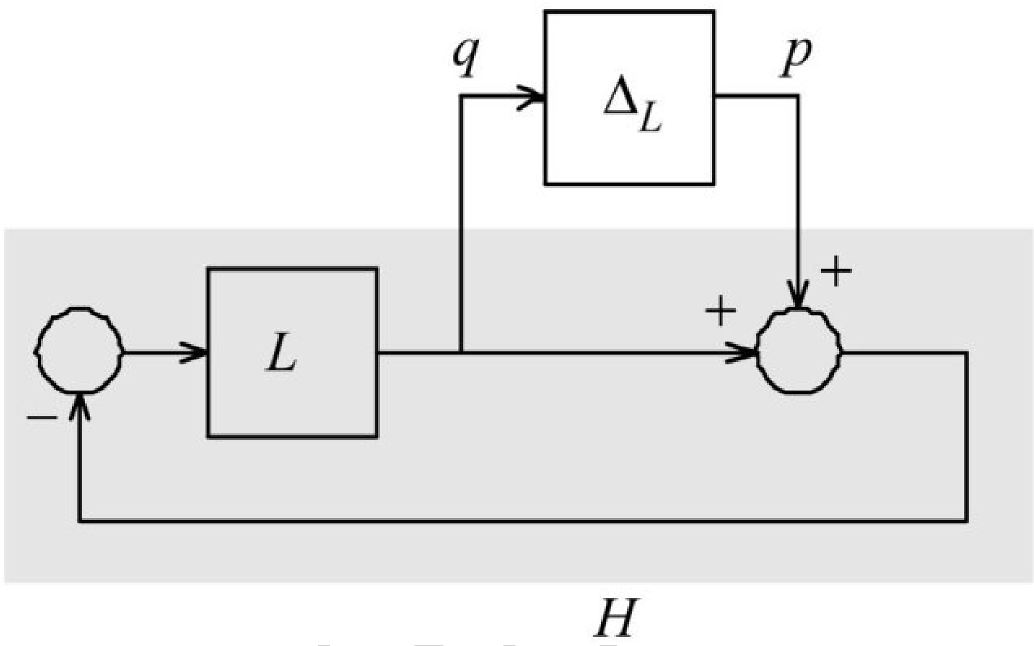

Figure 2. Perturbed feedback loop

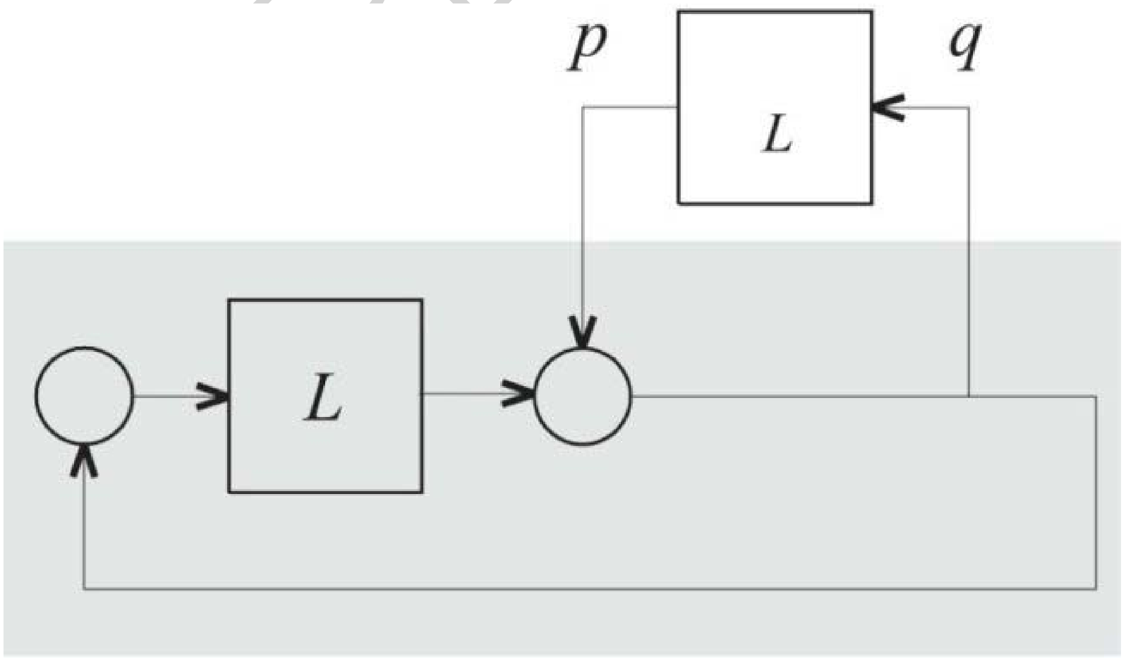

Figure 3. Alternative perturbation

The configuration of Figure 2 may be viewed as a feedback loop whose loop gain is the series connection of the block Δ_L and a block with transfer function $-T$, where

$$T = \frac{L}{1+L} \tag{10}$$

This transfer function is known as the “complementary sensitivity function” because $S+T=1$. By the small gain theorem, the closed-loop system is guaranteed to be stable if $\|\Delta_L T\| < 1$. By the sub-multiplicativity property of induced norms it follows that if

$$\|\Delta_L\| \cdot \|T\| < 1 \tag{11}$$

then $\|\Delta_L T\| < 1$ and, hence, the closed-loop system is guaranteed to be stable. It follows that for maximal robustness the norm $\|T\|$ of the complementary sensitivity function should be as small as possible.

This argument holds for any norm with the sub-multiplicativity property. A convenient choice is the ∞-norm that we met in the previous section.

The analysis so far points at the complementary sensitivity function as the critical system function for robustness. This is in conflict with the fundamental fact from feedback theory that the benefits that feedback brings are accomplished by making the loop gain *L* large. Inspection of Eq. (10), however, shows that if *L* is large then the complementary sensitivity *T* is close to 1, which is not particularly small. How can this be reconciled?

The answer to this question is that the small gain theorem leads to *sufficient* conditions, which guarantee stability but are not always necessary. Application of the small gain theorem in a slightly different framework may easily yield a different set of sufficient conditions. Consider for instance the block diagram of Figure 3. Note that the perturbation, denoted as $\Delta_{L^{-1}}$ for reasons that will soon become clear, is connected in a local feedback loop. A simple computation shows that the perturbation modifies the loop gain from *L* to

$$\frac{L}{1+\Delta_{L^{-1}}} \tag{12}$$

This expression becomes more transparent by noting that the perturbation modifies the inverse loop gain $1/L$ to

$$\frac{1+\Delta_{L^{-1}}}{L} \tag{13}$$

Thus, $\Delta_{L^{-1}}$ really is the relative perturbation of the inverse loop gain. There is no reason why we should not consider this, especially if we observe that the configuration of Figure 3 may be seen as the block $\Delta_{L^{-1}}$ connected in a feedback loop with another block with transfer function $-S = -1/(1+L)$, where *S* is the sensitivity function. By the small gain theorem, stability is guaranteed if $\|\Delta_{L^{-1}} \cdot S\| < 1$, which, in turn, certainly holds if

$$\|\Delta_{L^{-1}}\| \cdot \|S\| < 1 \tag{1}$$

Thus, the alternative analysis, based on perturbations of the inverse loop gain, leads to the sensitivity function *S* as a critical quantity for robustness. Much to our relief, inspection of Eq. (2) shows that if the loop gain *L* is large then *S* is small, which confirms the notion that a large loop gain implies good robustness.

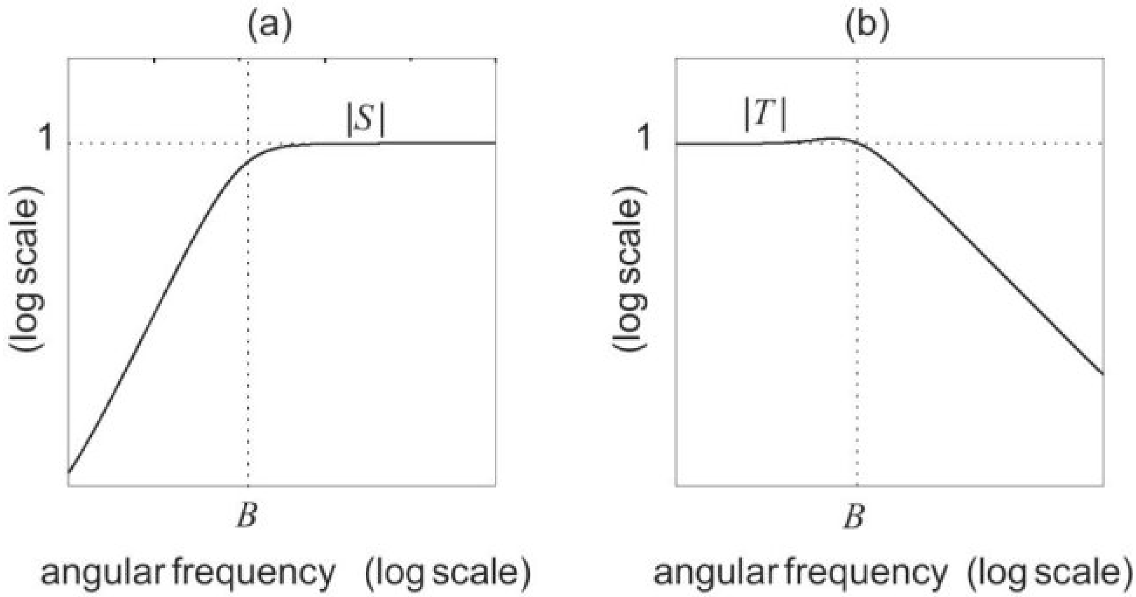

Figure 4. Ideal sensitivity functions

The nagging question remains: what is wrong with the previous analysis that led to the complementary sensitivity function *T* as the critical function? *S* and *T* cannot simultaneously be small, precisely because they are complementary: that is, add up to 1. The answer to this question is that the functions *S* and *T* are *both* critical for robustness, but never simultaneously, and in different, complementary frequency regions. The sensitivity function *S* is typically critical in the low frequency range, where the loop gain can be made large, the sensitivity to disturbances needs to be small, and the robustness with respect to parameter variations caused by load and other environmental changes needs to be good. The complementary sensitivity function *T* is critical—that is, needs to be small—at high frequencies. At high frequencies the loop gain *L*, and hence also *T*, should quickly drop off to very small values. This prevents waste of bandwidth, provides robustness against high frequency modeling uncertainty, and reduces the effect of measurement noise in the feedback loop.

Figure 4 shows ideal shapes for the sensitivity functions, with *S* small at low frequencies and *T* small at high frequencies. The actual critical frequency region is the crossover region, where the magnitude $|L|$ of the loop gain crosses over the zero dB line. The zero dB line separates the high and the low gain areas. For adequate stability, good time responses and sufficient robustness the Nyquist plot of the loop gain should stay away from the critical point -1 in the complex plane. The Nyquist plot approaches the critical point most closely for frequencies in the crossover region. If L gets very near to the critical point then both S and T peak to dangerously large values. (See also *Uncertainty Models for Robustness Analysis*.)

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Biographical Sketch

Huibert Kwakernaak was born in Rijswijk (Z.H.), the Netherlands, in 1937. He obtained his diploma in Engineering Physics from Delft University of Technology in 1955, an M.Sc. in Electrical Engineering in 1962 and a Ph.D. in Electrical Engineering in 1963, both from the University of California at Berkeley. He worked at Delft University of Technology from 1964, first in the Engineering Physics Department and later also in the Mathematics Department, until 1970 when he was appointed Professor in the Applied Mathematics Department of the University of Twente. He worked there until in 2002 he retired from his faculty position.

Huibert Kwakernaak's research interests are in linear control and systems theory. He is the co-author of three books, the best known of which is *Linear Optimal Control Systems,* with R. Sivan, Wiley-Interscience, 1972.

He is a Fellow of the IEEE and holds the Distinguished Service Award of IFAC. In 1994 he became Editor-in-Chief of *Automatica.* From 1995 until the end of 1999 he was the Scientific Director of the Dutch Institute of Systems and Control DISC, which is a national graduate school and research institute in the systems and control area in The Netherlands.