CONTROLLER DESIGN USING LINEAR MATRIX INEQUALITIES

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Summary

Using linear matrix inequalities for analysis and design of control systems is a relatively new field of research. Many control problems can be expressed in the form of linear matrix inequalities and solved with recently developed efficient convex optimization techniques. As an introduction to this area, this articles discusses the design of controllers with combined constraints on pole locations, the H_2-norm and the H_∞-norm of the closed-loop system. This approach is illustrated with a design example.

1. Introduction

It has been known for a long time that linear matrix inequalities play an important role in many control engineering problems. Recent developments in numerical algorithms for linear and convex optimization have renewed the interest in this field, and from the mid 1990s methods based on linear matrix inequalities (LMI) have become an active field of research.

Performance specifications for control systems are often expressed in terms of a quadratic performance index (the H_2-norm of a transfer function) or in terms of the largest gain across frequency (the H_∞-norm of a transfer function). Whereas the H_2-norm can be used to handle stochastic aspects and to trade control effort against regulation error, constraints on the H_∞-norm can be used to shape the frequency response of the loop transfer function and to include robustness specifications when system parameters are uncertain. H_2 and H_∞ conditions do however not give direct control over the shape of the transient response; specifications on damping, rise time and settling time are best expressed in terms of pole locations.

In many practical design problems, the above approaches – conditions on the H_2 or H_∞-norm and on pole locations – capture different aspects of the design, and it would be desirable to be able to use them simultaneously to find the best controller for a given problem. A tractable method for doing this was developed only recently: one can use linear matrix inequalities to combine these specifications into a single convex optimization problem.

This article introduces and illustrates the basic concepts of design methods based on linear matrix inequalities. It is organized as follows. In Section 2 a standard form of a convex optimization problem – minimization of a linear cost function under LMI constraints – is presented for which efficient solvers are available. It is then shown how the design specifications mentioned above can be expressed as LMI constraints.

In Section 3 it is discussed how the standard problem formulation can be used to search for controllers that achieve these specifications. In Section 4 a robust control problem is solved as a case study that illustrates the approach. Conclusions are drawn Section 5.

2. Design Specifications and Linear Matrix Inequalities

A linear matrix inequality (LMI) has the form

$$M(p) = M_0 + p_1 M_1 + \cdots + p_N M_N < 0, \qquad (1)$$

where $M_0, M_1, \ldots, M_N$ are given symmetric matrices, $p = [\, p_1\, p_2 \ldots p_N \,]^T$ is a column vector of real scalar variables (the decision variables), and the matrix inequality $M(p) < 0$ means that the left hand side is negative definite.

An important property of LMIs is that the set of all solutions p is convex.

Linear matrix inequalities can be used as constraints for the minimization problem

$$\min_{p} c^T p \quad \text{subject to} \quad M(p) < 0, \tag{2}$$

where the elements of the vector c in the linear cost function are weights on the individual decision variables. This problem is convex and can be solved by efficient, polynomial-time interior-point methods. Several LMI constraints can be combined into a single constraint of type (1). For example, $M_1(p) < 0$ and $M_2(p)$ are equivalent to the single LMI constraint

$$\begin{bmatrix} M_1(p) & 0 \\ 0 & M_2(p) \end{bmatrix} < 0. \tag{3}$$

The problem (2) is quite general, and a variety of problems can be reduced to this form. It has been known for a long time that many control problems can be expressed in this way. Already in 1971 the question was raised whether linear matrix inequalities can be exploited for numerical purposes in control engineering, but this happened only 20 years later when available computing power had dramatically increased and after efficient interior point methods for solving such problems had been developed in the 1980s. Control engineers became aware of this development in the early 1990s, standard software tools became available in the mid 1990s, and from then on LMI techniques developed into an active area of research. As an introduction to this field, in this article it is shown how the search for a controller that satisfies design specifications in terms of the H_2 and H_∞-norm and pole locations can be expressed in the form of (2).

2.1. Pole Region Assignment

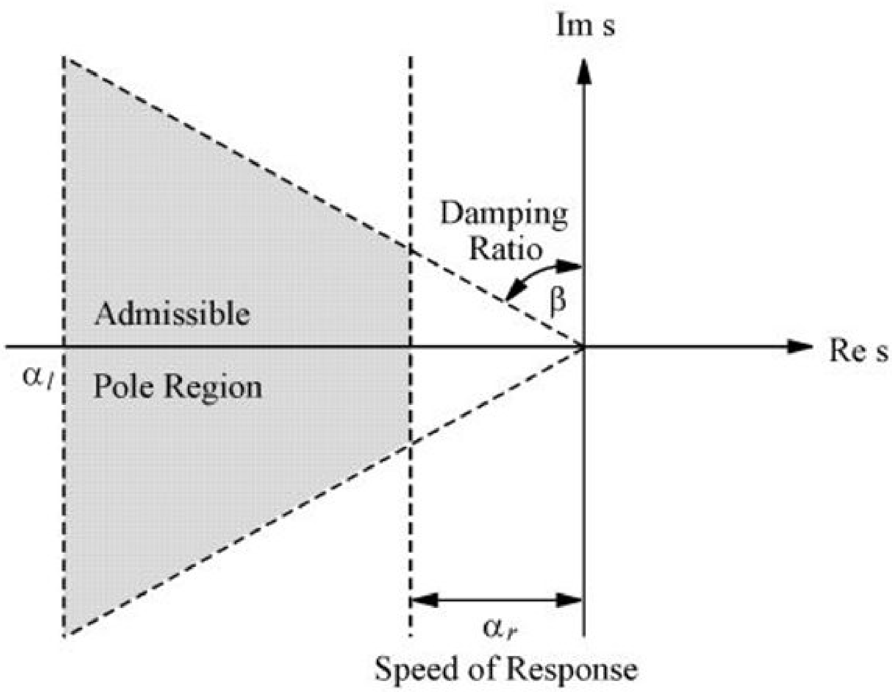

Figure 1: Desired pole region

It is well known that the dynamic properties of a linear, time-invariant system are determined by the location of the poles of its transfer function in the complex plane. For second order systems, there is a particularly simple relationship between the pole location and damping ratio, rise time and settling time. Standard methods are available to find a controller that places the poles in specified locations in the complex plane. However, it is not obvious where exactly the poles of the closed-loop system should be located to achieve good performance: the required control effort is larger when the poles are moved far away from their original locations, and in the presence of transfer function zeros these also have an effect on the response. In practice the designer often works on specifications that include a minimum damping ratio and a minimum speed of response, and it is desired to find the best controller (in terms of a suitable performance index) that satisfies these constraints. The constraints can be expressed as a region in the complex plane where the closed-loop poles should be located. A typical pole region is shown in Figure 1.

The condition that the poles of a system are located within a given region in the complex plane can be formulated as an LMI constraint. Here is a simple example: the homogenous system

$$\dot{x}(t) = Ax(t) \tag{4}$$

is stable if and only if the matrix A has all eigenvalues in the left half plane, which in turn is true if and only if there exists a positive definite, symmetric matrix P that satisfies

$$PA^T + AP < 0. \tag{5}$$

This result was established by the Russian mathematician Lyapunov more than 100 years ago, and the inequality (5) is referred to as Lyapunov inequality. This inequality is linear in the matrix variable P, and one can use efficient LMI solvers to search for solutions. It is straightforward to rewrite (5) in the standard form (1) of an LMI. To see this, assume that A is a 2 by 2 matrix and write the symmetric matrix variable P as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = p_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Substitution of the right hand side for P in (5) yields an LMI in the form of (1) with three decision variables p_1, p_2 and p_3. Available software tools operate directly on matrix variables, so that it is usually not necessary to carry out this transformation.

The LMI (5) represents a necessary and sufficient condition for the matrix A to have all eigenvalues in the left half plane. It is possible to generalize this result: one can express an arbitrary region $\mathcal{D}$ in the complex plane such as the one shown in Figure 1 (as long as it is convex and symmetric about the real axis) in terms of two matrices $L = L^T$ and M as the set of all complex numbers that satisfy an LMI constraint

$$\mathcal{D} = \{s \in \mathbb{C} : L + Ms + M^T \overline{s} < 0\}, \quad (6)$$

where $\overline{s}$ denotes the complex conjugate of s. Such a region is called and *LMI region*. One can show that a necessary and sufficient condition for a matrix A to have all eigenvalues in $\mathcal{D}$ is the existence of a positive definite, symmetric matrix P that satisfies

$$L \otimes P + M \otimes (AP) + M^T \otimes (AP)^T < 0. \quad (7)$$

The symbol $\otimes$ stands for the Kronecker product: if M is a 2 by 2 matrix then

$$M \otimes P = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \otimes P = \begin{bmatrix} m_{11}P & m_{12}P \\ m_{21}P & m_{22}P \end{bmatrix}.$$

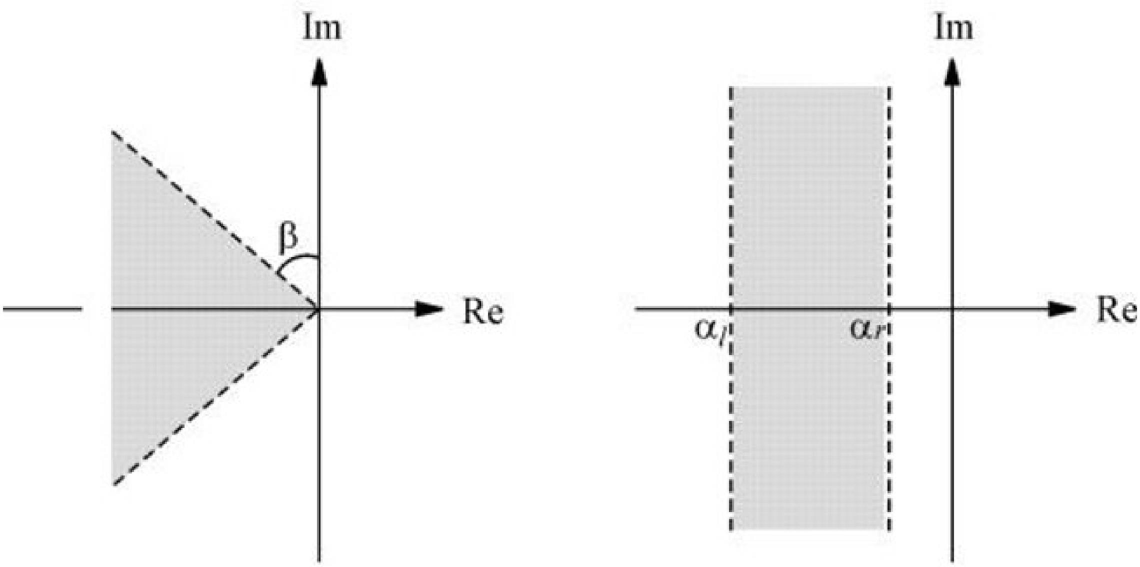

Figure 2: LMI regions

Thus if P is also 2 by 2 the Kronecker product is 4 by 4. Inequality (7) is an LMI in the matrix variable P. It is easy to see that the Lyapunov inequality (5) is obtained as a special case of (7) with $L = 0$ and $M = 1$ and $\mathcal{D}$ as the left half plane. One can verify that the conic sector shown in Figure 2 is an LMI region with

$$L_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_c = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix},$$

where $\theta = 90^0 - \beta$, and the vertical strip is an LMI region with

$$L_v = \begin{bmatrix} 2\alpha_l & 0 \\ 0 & -2\alpha_r \end{bmatrix}, \quad M_v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The intersection of both regions is the pole region shown in Figure 1; it can be represented as an LMI region by combining the two constraints as in (3) with

$$L = \begin{bmatrix} L_c & 0 \\ 0 & L_v \end{bmatrix}, \quad M = \begin{bmatrix} M_c & 0 \\ 0 & M_v \end{bmatrix}.$$

2.2. H_2 Performance

In this and the following section, a state space model

$$\dot{x}(t) = Ax(t) + Bw(t)$$

$$z(t) = Cx(t) + Dw(t) \quad (8)$$

is considered, where the signal vector $w(t)$ represents external inputs such as reference input, disturbances or noise, and the signal vector $z(t)$ contains fictitious outputs that are used to assess performance.

Let $G(s)$ denote the transfer function matrix from w to z. In this subsection, it is assumed that A is stable and $D = 0$, thus

$$G(s) = C(sI - A)^{-1}B.$$

The H_2-norm $\|G\|_2$ of $G(s)$ is defined by

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left[G^*(j\omega)G(j\omega)\right]d\omega,$$

where G^* denotes the complex conjugate transpose of G. An equivalent definition in time domain is

$$\|G\|_2^2 = \int_0^{\infty} \text{trace}\, g^T(t)g(t)\,dt,$$

where $g(t)$ is the impulse response matrix of the system. The H_2-norm can be interpreted as the root mean square output power $E[z^T z]^{1/2}$ if the input w is a vector white noise process with autocovariance matrix $E[ww^T] = I$. All control schemes that minimize quadratic performance indices, such as LQG, LQR, and Kalman-Bucy filtering, minimize the H_2-norm of a transfer function matrix. In this subsection it is shown how a constraint on this norm can be expressed in terms of linear matrix inequalities.

The key for this is the well known fact that

$$\|G\|_2^2 = \text{trace } CP_0C^T ,$$

where P_0 is the controllability Gramian that satisfies

$$AP_0 + P_0A^T + BB^T = 0 .$$

It is also true that

$$\|G\|_2^2 < \text{trace } CPC^T$$

for any P that satisfies

$$AP + PA^T + BB^T < 0 \qquad (9)$$

because $P > P_0$. This fact is used to express an upper bound on the H_2-norm in the form of a linear matrix inequality: $\|G\|_2 < \nu$ if and only if there exists a matrix $P > 0$ that satisfies (9) and

$$\text{trace } CPC^T < \nu^2. \qquad (10)$$

In Section 3 it is shown how one can search over controllers that result in the closed-loop system satisfying the above constraint. The system matrices A, B and C will then be closed-loop system matrices that contain controller parameters, and the expression trace CPC^T will be nonlinear in these parameters. The following equivalent formulation of the above result is then used instead. Introduce a new symmetric matrix variable W (which is used as a slack variable), then $\|G\|_2 < \nu$ if and only if there exist symmetric matrices P and W that satisfy (9) and

$$\begin{bmatrix} W & CP \\ PC^T & P \end{bmatrix} > 0 \text{ and } \text{trace}\, W < \nu^2. \qquad (11)$$

To see that (11) is equivalent to (10), note that

$$\begin{bmatrix} M & L \\ L^T & N \end{bmatrix} > 0,$$

where $M = M^T$ and $N = N^T$, is equivalent to

$$N > 0 \quad \text{and} \quad M - LN^{-1}L^T > 0.$$

This fact is frequently used to convert nonlinear inequalities into LMI form; the term $M - LN^{-1}L^T$ is the Schur complement with respect to N. In the above case, this leads to $W > CPC^T$ and consequently to trace $W > \mathrm{trace}\, CPC^\mathrm{T}$, from which the equivalence of (10) and (11) follows.

2.3. H_∞ Performance

Consider again the linear model (8) with *A* stable and let $G(s)$ denote the transfer function matrix from *w* to *z*. The H_∞- norm of $G(s)$ is defined as

$$\|G\|_\infty = \sup_\omega \sigma_{\max}(G(j\omega)),$$

where $\sigma_{\max}(G)$ denotes the largest singular value of G. For practical purposes, the supremum "sup" of a function is the same as its maximum value (it may approach its maximum as a limit). The H_∞-norm can be interpreted as the maximum transfer function matrix gain on the $j\omega$ axis (the largest gain across frequency in the singular value norm). One can use constraints on the H_∞-norm to express frequency domain specifications such as bandwidth, low frequency gain or roll-off, and to incorporate robustness issues into the design. An example for the latter is given in Section 4.

An equivalent definition of the H_∞-norm is

$$\|G\|_\infty^2 = \sup_{w \neq 0} \frac{\int_0^\infty z^T(t)\, z(t)\, dt}{\int_0^\infty w^T(t) w(t)\, dt},$$

where it is assumed that $x(0) = 0$. Therefore, $\|G\|_\infty$ is the maximum possible gain in signal energy. This fact can be used to express constraints on the H_∞-norm in terms of linear matrix inequalities. From the above it follows that $\|G\|_\infty < \gamma$ is equivalent to

$$\int_0^\infty \left(z^T(t)\; z(t) - \gamma^2 w^T(t) w(t) \right) dt < 0$$

holding true for all square integrable $w(t)$. Now introduce a Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$. Because $x(0) = x(\infty) = 0,$ the constraint $\|G\|_\infty < \gamma$ is then enforced by the existence of a matrix $P = P^T > 0$ such that

$$\frac{dV(x)}{dt} + \frac{1}{\gamma} z^T(t) z(t) - \gamma w^T(t) w(t) < 0 \tag{12}$$

for all $x(t), w(t);$ this can be seen by integrating (12) from $t = 0$ to $t = \infty$. To turn (12) into a linear matrix inequality, substitute

$$\frac{dV(x)}{dt} = x^T (A^T P + PA) x + x^T PBw + w^T B^T P x$$

and $z = Cx + Dw$ in (12) to obtain

$$\begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} A^T P + PA + \frac{1}{\gamma} C^T C & PB + \frac{1}{\gamma} C^T D \\ B^T P + \frac{1}{\gamma} D^T C & -\gamma I + \frac{1}{\gamma} D^T D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0.$$

For $\|G\|_\infty < \gamma$ the above must hold for all x and w, thus the block matrix must be negative definite. This condition can be rewritten as

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

and using the Schur complement (see Section 2) it follows that $\|G\|_\infty < \gamma$ if there exists a positive definite, symmetric matrix P that satisfies the linear matrix inequality

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \tag{13}$$

It can be shown that this is not only a sufficient but also a necessary condition for $\|G\|_\infty < \gamma$.

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Bibliography

Boyd, S., L.El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear matrix inequalities in system and control theory.* Vol.15 of Studies in Applied Mathematics, SIAM, Philadelphia, PA, USA.[A comprehensive overview about linear matrix inequalities in control is given]

El Ghaoui. L. and S.I. Niculescu (ed.) (2000). *Advances in linear matrix inequality methods in control.* Advances in Design and Control, SIAM, Philadelphia, PA, USA. [An update on recent advances in LMI techniques is given in this publication]

Gahinet, P.(1996). Explicit controller formulas for LMI-based H_∞ synthesis. *Automatica* **32** (7) 1007-1014. [The relation between LMI-based controllers and Riccati-based controllers is discussed]

Gahinet, P., A. Nemirovski, A.J.Laub and M.Chilali (1995). *LMI control toolbox for use with Matlab.* The Mathoworks Inc., Natick, MA, USA. [LMI software tools are documented]

Nesterov, Y. and A. Nemirovski (1994). *Interior-point polynomial methods in convex programming.* Vol.13 of studies in Applied Mathematics, SIAM, Philadelphia, PA, USA. [The interior-point methods employed for solving LMI problems are described]

Scherer, C., P. Gahinet and M.Chilali (1997). Multiobjective output feedback control via LMI optimization. *IEEE Trans. Aut. Control* **42** (7), 896-911. [Output feedback controller design with pole region constraints and constraints on the H_2 and H_∞ -norm is discussed]

Skelton, R., T. Iwasaki and K. Grigoriadis (1998). *A unified approach to linear control design.* Taylor & Francis, London. [Design of LMI-based controllers with dynamic order lower than the plant is considered]

Werner, H., P. Korba and T.C. Yang (2003). Robust tuning of Power System Stabilizers using LMI Techniques. *IEEE Transactions Contr. Systems Technology* (to appear). [The case study in Section 4 is taken from this publication]

Willems, J.C. (1971). Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Aut. Control* **16** (6), 621-634.[The role of linear matrix inequalities in quadratic optimal control was discussed]

Biographical Sketch

Herbert Werner received the Dipl.-Ing. degree from the Ruhr University Bochum, Germany, the MPhil degree from the University of Strathclyde, UK, and the Doctor of Engineering degree from the Tokyo Institute of Technology, Japan, in 1989, 1991 and 1995, respectively. From 1995 to 1998 he was with the Control Engineering Lab at the Ruhr University Bochum, and from 1999 to 2002 with the Control Systems Centre at UMIST, UK, where he was a Senior Lecturer. In 2002 he was appointed head of the Control Engineering Institute of the Technical University Hamburg-Harburg, Germany. His research interests include linear system theory, robust and nonlinear control, and system identification.