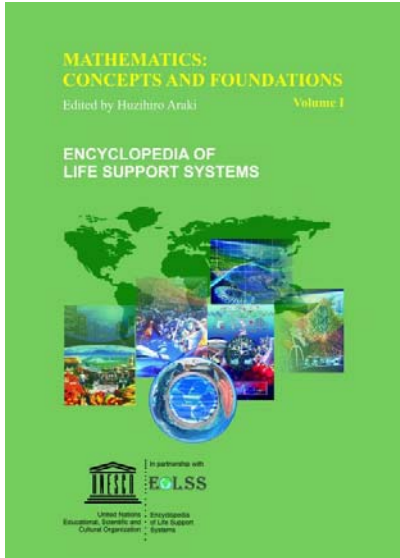


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Phokion G. Kolaitis, *Computer Science Department, University of California, Santa Cruz, CA 95064, USA*

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Kit Fine, *Professor of Philosophy and Mathematics, New York University, USA*

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University of Paris 7 and Laboratoire Jacques Louis Lions (Univ. Paris 6), France

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Fabrice Bethuel, Universite Pierre et Marie Curie, Paris, France

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Louis Boutet de Monvel, *Universit' e Pierre et Marie Curie, France*

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J. J. Duistermaat, *Department of Mathematics, Utrecht University, The Netherlands*

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Claude Bardos, *University Denis Diderot, France*

Louis Boutet de Monvel, *Universit' e Pierre et Marie Curie, France*

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Kazuo Murota, *University of Tokyo, Tokyo, Japan*

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Hikoe Enomoto, *Hiroshima University, Higashi-Hiroshima, Japan*

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Takeshi Tokuyama, *Tohoku University, Sendai, Japan*

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Osamu Watanabe, *Tokyo Institute of Technology, Tokyo, Japan*

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Toshihide Ibaraki, *Kyoto University, Kyoto, Japan*

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