GRAVIMETRY

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Contents

Summary

After a brief survey of the Newton's theory of the potential and of the gravitational and gravity forces, a discussion of facts of the main problems of the gravimetry follows. It is the question of the theory of the geoid, actually quasigeoid, level surfaces and deviations from the consequent deflections of the vertical. The basic methods of the measurements of the gravity forces are examined as well as their reductions. Then the

applications of the gravimetry in the practice are introduced, for example the heights computation and different methods of leveling.

1. Introduction

Gravimetry as an independent scientific branch began to evolve only at the end of the 19th century. However, its roots go back to the times of the celebrated pioneers of classical mechanics, G. Galileo and I. Newton (16th and 18th centuries). A number of outstanding 18th and 19th century scientists (e.g. A.C. Clairaut, G. Stokes, H. Bruns, F.R. Helmert) worked on various fundamental gravimetric problems concerned in particular with the application of the theory of potential to research into the Earth's body. Nevertheless, the term gravimetry (from Latin gravis=heavy and the Greek metrein=to measure) only became established in science in the 20th century. The name does not fully reflect the nature of the subject. A gravimetrician not only must measure gravity, search for new methods of measurements and construct gravimetric instruments, but also has to solve many fundamental gravimetric problems theoretically and practically.

Among the most important problems of contemporary gravimetry it is the study of the figure and the dimensions of the Earth's body (the geoid and its external gravity field).

Treatment of the important problem of the equilibrium of the Earth's crust (the problem of isostasy) was another significant task of gravimetricians. However, the first observations of the artificial satellites already showed that isostasy does not exist.

The Earth's crust is also acted upon by attractive forces due to the masses of celestial bodies, of which the moon and the sun have the largest effects. These disturbing forces result in periodical movements in the Earth's crust called the tides of the solid crust. Similarly, the same forces act on the seas and oceans, creating the sea tides (generally known as tides and ebbs), and on the gaseous shell of the Earth causing atmospheric tides, caused mostly by the influence of the thermal radiation of the Sun.

Of great economic importance is the problem of applying gravimetry to surveying and investigating deposits of useful minerals and raw materials. From this list of some of more important gravimetric problems it can be seen that gravimetry is closely related to geodesy and geology. As compared with geodesy which uses geometric methods to investigate the figure of the Earth, gravimetry employs physical methods for the same problem. Geology uses applied gravimetry in the investigation of the geological structure of the upper parts of the Earth's crust, and gravimetric methods are employed in prospecting. Gravimetry is also intimately connected with physics, mathematics, astronomy and other related sciences.

The first rather inaccurate measurements of the acceleration of gravity were made by Galileo Galilei (1564-1642), who observed the path of freely falling bodies. According to the law of free fall, which he discovered in about 1590, the length of the path of a body falling from its initial rest position in the first second is equal to half of the value of acceleration of gravity at the point of observation.

The founder of the science of the attraction of masses (gravitation) is Sir Isaac Newton (1642-1727), who was the first to publish the law of universal gravitation in 1687.

Apart of these two great men, many other important scientists have a place in the history of gravimetry. A contemporary of Newton, Ch. Huygens (1629-1695) dealt with the problem of the Earth's shape as well. He also elaborated the theory of the physical pendulum in 1673.

The disagreements between the theoretical considerations of Newton and Huygens were explained by A.C. Clairaut (1713-1765), who also showed how the flattening of the Earth could be computed from gravimetric observations. C. Maclaurin, P.S. Laplace, K. Jacobi, A.M. Lyapunov, A.M. Legendre, G. Stokes, G. Green, F.R. Helmert, J. Bruns, and many others dealt with various theoretical problems of gravimetry in the 17th, 18th, 19th and 20th centuries.

Stokes' study (1819-1903) was especially important. He indicated that the figure of the Earth (the geoid) could be derived if the distribution of the acceleration of gravity was known over the whole surface of the Earth. His theoretical results have, in fact, only been exploited in the 1950s as sufficient gravity data from the whole world became available. Many scientists of the 19th and 20th century dealt with the figure of the geoid: L. Tanni, M.S. Molodenskii, P. Pizzetti, and many others.

In the 17th and 18th centuries, Jacques, Jean and Daniel Bernoulli elaborated the theory of the physical pendulum. Various modifications of reversible pendulums were used for measurements of the acceleration of gravity. The first pendulum measurements of the acceleration of gravity on seas and oceans were carried out in 1928 by F.A. Meinesz.

2. Newton's Theory of Potential

Attractive forces called gravitational forces operate between the Earth and an arbitrary body. They cause every body to have weight, and if a body has an opportunity to move, it falls towards the Earth with a given acceleration. These forces govern the motions of celestial bodies. From the ordered movements of the planets around the sun, Kepler derived his three laws. From the latter, Newton in turn formulated the law of universal gravitation. This law is as follows:

The force with which two bodies of masses m_1 and m_2 attract each other is directly proportional to the product of both masses and inversely proportional to the square of the distance between them:

$$F = G\frac{m_1 . m_2}{r^2}, \tag{2.1}$$

where *the gravitational constant*

$G = 6.670.10^{-11} \mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$.

Therefore, the bodies form gravitational field in space which can be described by means of the intensity of the field or by means of the gravitational potential. The intensity of the gravitational field is understood to be the force which acts on a unit mass at a given point. The gravitational potential at a point is the amount of work required to be done to move a unit mass from infinity (point of zero potential) to that point. In this case the potential is defined as a force function in mechanics. It should not be mistaken for the potential energy. The geometric locus of points with the same potential is an equipotential surface.

If a celestial body is sufficiently small relative to the distance from the Earth, we can consider the gravitational effect of this body as effect of a point mass. Let us have a set of point masses. The coordinates of the pole P are (ξ,η,ζ). The coordinates of the variable point P_i are (x_i, y_i, z_i). The gravitational effect of this set is given as a sum effects for individual mass points determined by Eq. (2.1). We usually put $m_P = 1$. For $G = 1$ we have

$$\vec{F} = \sum_{i=1}^{n} \frac{m_{A_i}}{r_i^3} \vec{r}_i \,. \tag{2.2}$$

The following expression holds for the components of force $\vec{F}$ in the direction of the coordinate axes:

$$X = \sum_{i=1}^{n} m_{A_i} r_i^{-3}(x_i - \xi),\ Y = \sum_{i=1}^{n} m_{A_i} r_i^{-3}(y_i - \eta),\ Z = \sum_{i=1}^{n} m_{A_i} r_i^{-3}(z_i - \zeta). \tag{2.3}$$

The components X, Y, Z may be considered as partial derivatives of the potential of the point masses, U, i.e.,

$$X = \partial\partial U/\partial\xi,\ Y = \partial U/\partial\eta,\ Z = \partial U/\partial\zeta \,.$$

The potential U is determined from the equation

$$U = \sum_{i=1}^{n} r_i^{-1}.m_{A_i} \,. \tag{2.4}$$

The potential U is called *volume potential*. Eq.(2.4) can be written in the form

$$U = \iiint_{\tau} \rho \frac{d\tau}{r} \,, \tag{2.5}$$

where ρ is the density of the body and τ is the region of integration.

The first derivatives of the function U can be considered in the form

$$\frac{\partial U}{\partial \xi} = \iiint_{\tau} \rho \frac{x-\xi}{r^3} d\tau = \iiint_{\tau} \frac{\rho}{r^2} \cos(r,x) d\tau. \tag{2.6}$$

The same applies to the components Y and Z. If the density ρ is capable of integration and limited, then the volume potential U and its derivatives are continuous functions in the whole region and they are limited, too.

The second derivatives of the volume potential

$$\Delta U = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2}, \tag{2.7}$$

where

$\Delta U_i = -4\pi\rho(P)$, P within τ, (Poisson equation),
$\Delta U_e = 0$, P outside τ, (Laplace equation),

and

$$\Delta U_e - \Delta U_i = 4\pi\rho_i. \tag{2.8}$$

Further (*Gauss formula*)

$$M = -\frac{1}{4\pi} \iint \sigma \frac{\partial U}{\partial n} d\sigma, \tag{2.9}$$

where M is the total mass of the body, limited by the surface σ, n is the outer normal to the surface σ.

At improper (infinitely distant) points

$$\lim_{r \to \infty} rU = M.$$

The function U is regular at infinity.

From the theory of the electromagnetic field was adopted the concept of the surface potentials, the potential of the *single layer* and *the potential of the double layer.*

The potential of the double layer W is defined by the equations

$$W = \iint_{\sigma} \nu \frac{\cos\varphi}{r^2} d\sigma = -\iint_{\sigma} \nu \frac{\partial}{\partial n}\left(\frac{1}{r}\right) d\sigma, \tag{2.10}$$

where n is the outer normal to the surface σ, φ is the angle between the radius vector and the external normal at the variable point.

The potential of the double layer is a discontinuous function, if the pole P passes through the surface σ. If the pole P approaches σ in infinity, from inside or outside, we obtain the limiting values W_i and W_e respectively

$$\begin{aligned} &W_i(P) = W(P_o) + 2\pi\nu(P_o), \\ &W_e(P) = W(P_o) - 2\pi\nu(P_o), \\ &W(P) = \iint_\sigma \nu \frac{\cos\varphi_o}{r_o^2} d\sigma, \end{aligned} \tag{2.11}$$

where φ_o is the angle between the external normal at point A to the surface σ and radius vector r_o, $r_o = \overline{P_o A}$.

Potential of a single layer V is defined by the equation

$$V = \iint_\sigma \frac{\mu}{r} d\sigma \tag{2.12}$$

for the density μ. We can also write the expressions for the normal derivatives of the single layer potential. If the point P_o approximates the point P_o on the surface σ from inside or outside, we get

$$\begin{aligned} &\frac{\partial V_i}{\partial n_i} = \partial V / \partial n_o + 2\pi\mu(P_o), \\ &\frac{\partial V_e}{\partial n_e} = \partial V / \partial n_o - 2\pi\mu(P_o), \\ &\frac{\partial V}{\partial n_o} = \iint_\sigma \nu \frac{\cos\psi_o}{r_o^2} d\sigma, \end{aligned} \tag{2.13}$$

where the angle ψ_o is the angle between the outer normal n at the point P_o on the surface σ and the radius vector $r = \overline{P_o A}$.

The considerable importance of equations (2.10) - (2.13) was pointed out by Fredholm. They can be applied in solving the boundary problems of the potential theory by means of integral equations.

Hitherto, we have assumed that the volume and surface potentials are related to bodies and surfaces which are stationary. If the body is rotating around a fixed axis of rotation with a constant angular velocity ω, then the force $\vec{F} = \omega^2 \vec{\rho}$ and the potential $U = \frac{1}{2}\omega^2 \rho^2$ appear.

The gravitational potential V is due to the action of the gravitational forces, and the potential U is due to the centrifugal force. The sum of these potentials

$$W = U + V \tag{2.14}$$

is potential of the gravity, or the *gravity potential*.

3. Potential of Some Simple Formations, Approximate in Shape to the Figure of the Earth

Let us consider a sphere σ, centred at $O(0,0,0)$, with radius R. Point P is outside the sphere, its coordinate are $P(\rho,0,0)$. For the external pole, the gravitational potential is given by the relation

$$V_e = \frac{4}{3}\pi G\kappa \frac{R^3}{\rho}, \ \ \rho \geq R, \tag{3.1}$$

and for internal point by the equation

$$V_i = \frac{2}{3}\pi G\kappa (3R^2 - \rho^2), \ \ \rho \leq R. \tag{3.2}$$

Equipotential rotational ellipsoid is usually put for the mathematical model of the Earth. According to Pizzetti, its gravity potential, that is the sum of the gravitational potential and potential of the centrifugal force, is given by the relation

$$W = (\alpha_o + \beta_o)A_o - \alpha_o\{[B - B_o - \frac{b'r}{a'^2}C + \frac{b^2}{a^2}C_o]r^2 + b'^2C\}, \tag{3.3}$$

where α_o, β_o are constants, A, B, C are Jacobi's integrals.

The gravity γ of this body can be determined by taking the derivative of this equation in the direction of the outer normal:

$$\gamma = \sqrt{(\frac{\partial W}{h_u \partial u})^2 + (\frac{\partial W}{h_s \partial s})^2}. \tag{3.4}$$

The influence of the second term can be mostly neglected. Thus, we can write

$$\gamma \sim \frac{\partial W}{h_u \partial u},$$

$$\gamma = -\frac{2\alpha_o}{\sqrt{Q}}[(B - B_o)\cos^2 u + C - (C - \frac{b^2}{a^2}C_o)\cos^2 u] - \frac{2\beta_o}{a'^2 b'\sqrt{Q}}, \tag{3.5}$$

where u is the reduced latitude.

Originally, according to the recommendation of the International Union of Geodesy and Geophysics of Luzern 1967, the physical and geometrical properties of the level rotational ellipsoid were fully determined by four parameters: major semiaxis a, the product of the Earth mass M and gravitational constant G, Stoke's coefficient J_2 and the angular velocity of the Earth's rotation ω. Later, according to Browars proposal, instead of the value a, the mean value of the potential of the world seas W_o was accepted.

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Bibliography

M.Pick, J.Pícha, V.Vyskočil (1973) Theory of the Earth Gravity Field. Elsevier . [Textbook of the modern gravimetry with particular reference to Molodensky's theory.]

M.S.Molodenskij (1945) Osnovnyje voprosy geodězičeskoj gravimetrii. Trudy CNIIGAiK, Vyp. 42, Moscow. [Basic problems of geodetic gravimetry. The English version by the Office of Technical Services U.S. Washington D.C.]

M.S.Molodenskij, V.F.Eremeev, M.I.Yurkina (----) Methods for Study of the External Gravitational Field and F???. — The survey of the modern Molodensky's school.

P.Vaníček, E.Krakiwsky (1986) Geodesy the concepts. Elsevier Amsterdam. [Modern textbook of the gravimetry. A good survey of the whole modern geodesy.]

W.A.Heiskanen, H.Moritz (1967) Physical geodesy. Freeman. [Classic monograph of the physical geodesy. The modern ways of solution are hinted only.]

Biographical Sketch

Miloš Pick, born in Luže, Czech Republic. He is a professor of Higher Geodesy, Czech Technical University, Prague and consultant, Geophysical Institute, Czech Academy of Sciences, Prague.

1945 – 1950 : student of geodesy, Czech Technical University, Prague

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1960 – 1970 :director, Geophysical Institute, Czechoslovak Academy of Sciences, Prague

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1967 — invited series of lectures of Newtonian Potential Theory, Charles University, Prague

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