

## A VIEW OF MATHEMATICS

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### Summary

Mathematics is the backbone of modern science and a remarkably efficient source of new concepts and tools to understand the "reality" in which we participate. It plays a basic role in the great new theories of physics of the XXth century such as general relativity, and quantum mechanics.

The nature and inner workings of this mental activity are often misunderstood or simply ignored even among scientists of other disciplines. They usually only make use of rudimentary mathematical tools that were already known in the XIXth century and miss completely the strength and depth of the constant evolution of our mathematical concepts and tools.

The author was asked to write a general introduction to Mathematics which ended up as a rather personal point of view rather than producing the usual endless litany “X did this and Y did that”. The evolution of the concept of “space” in mathematics serves as a unifying theme starting from some of its historical roots and going towards more recent developments in which the author has been more or less directly involved.

## 1. The Unity of Mathematics

It might be tempting at first to view mathematics as the union of separate parts such as Geometry, Algebra, Analysis, Number theory etc... where the first is dominated by the understanding of the concept of “space”, the second by the art of manipulating “symbols”, the next by the access to “infinity” and the “continuum” etc...

This however does not do justice to one of the most essential features of the mathematical world, namely that it is virtually impossible to isolate any of the above parts from the others without depriving them from their essence. In that way the corpus of mathematics does resemble a biological entity which can only survive as a whole and would perish if separated into disjoint pieces.

The first embryo of mental picture of the mathematical world one can start from is that of a network of bewildering complexity between basic concepts. These basic concepts themselves are quite simple and are the result of a long process of “distillation” in the alembic of the human thought.

Where a dictionary proceeds in a circular manner, defining a word by reference to another, the basic concepts of mathematics are infinitely closer to an “indecomposable element”, a kind of “elementary particle” of thought with a minimal amount of ambiguity in their definition.

This is so for instance for the natural numbers where the number 3 stands for that quality which is common to all sets with three elements. That means sets which become empty exactly after we remove one of its elements, then remove another and then remove another. In that way it becomes independent of the symbol 3 which is just a useful device to encode the number.

Whereas the letters we use to encode numbers are dependent of the sociological and historical accidents that are behind the evolution of any language, the mathematical concept of number and even the specificity of a particular number such as 17 are totally independent of these accidents.

The “purity” of this simplest mathematical concept has been used by Hans Freudenthal to design a language for cosmic communication which he called “Lincos”.

The scientific life of mathematicians can be pictured as a trip inside the geography of the “mathematical reality” which they unveil gradually in their own private mental frame.

It often begins by an act of rebellion with respect to the existing dogmatic description of

that reality that one will find in existing books. The young “would be mathematicians” realize in their own mind that their perception of the mathematical world captures some features which do not quite fit with the existing dogma.

This first act is often due in most cases to ignorance but it allows one to free oneself from the reverence to authority by relying on one’s intuition provided it is backed up by actual proofs. Once mathematicians get to really know, in an original and “personal” manner, a small part of the mathematical world, as esoteric as it can look at first (my starting point was localization of roots of polynomials), their trip can really start. It is of course vital all along not to break the “fil d’arianne” which allows to constantly keep a fresh eye on whatever one will encounter along the way, and also to go back to the source if one feels lost at times...

It is also vital to always keep moving. The risk otherwise is to confine oneself in a relatively small area of extreme technical specialization, thus shrinking one’s perception of the mathematical world and of its bewildering diversity.

The really *fundamental point* in that respect is that while so many mathematicians have been spending their entire scientific life exploring that world they all agree on its contours and on its connexity: whatever the origin of one’s itinerary, one day or another if one walks long enough, one is bound to reach a well known town *i.e.* for instance to meet elliptic functions, modular forms, zeta functions. “All roads lead to Rome” and the mathematical world is “connected”.

In other words there is just “one” mathematical world, whose exploration is the task of all mathematicians and they are all in the same boat somehow.

Moreover exactly as the existence of the external material reality seems undeniable but is in fact only justified by the coherence and consensus of our perceptions, the existence of the mathematical reality stems from its coherence and from the consensus of the findings of mathematicians.

The fact that proofs are a necessary ingredient of a mathematical theory implies a much more reliable form of “consensus” than in many other intellectual or scientific disciplines. It has so far been strong enough to avoid the formation of large gatherings of researchers around some “religious like” scientific dogma imposed with sociological imperialism.

Most mathematicians adopt a pragmatic attitude and see themselves as the explorers of this “mathematical world” whose existence they don’t have any wish to question, and whose structure they uncover by a mixture of intuition, not so foreign from “poetical desire”(as emphasized by the French poet Paul Valery), and of a great deal of rationality requiring intense periods of concentration.

Each generation builds a “mental picture” of its own understanding of this world and constructs more and more penetrating mental tools to explore previously hidden aspects of that reality.

Where things get really interesting is when unexpected bridges emerge between parts of the mathematical world that were previously believed to be very far remote from each other in the natural mental picture that a generation had elaborated. At that point one gets the feeling that a sudden wind has blown out the fog that was hiding parts of a beautiful landscape.

We shall see at the end of this chapter one recent instance of such a bridge. Before doing that we will take the concept of “space” as a guideline to take the reader through a guided tour leading to the edge of the actual evolution of this concept both in algebraic geometry and in noncommutative geometry.

We shall also review some of the “fundamental” tools that are at our disposal nowadays such as “positivity”, “cohomology”, “calculus”, “Abelian categories” and most of all “symmetries” which will be a recurrent theme in the three different parts of this text.

It is clearly impossible to give a “panorama” of the whole of mathematics in a reasonable amount of write up. But it is perfectly possible, by choosing a precise theme, to show the frontier of certain fundamental concepts which play a central role in mathematics and are still actively evolving.

The concept of “space” is sufficiently versatile to be an ideal theme to display this active evolution and we shall confront the mathematical concept of space with physics and more precisely with what Quantum Field Theory teaches us and try to explain several of the open questions and recent findings in this area.

## 2. The Concept of Space

The mental pictures of geometry are easy to create by exploiting the visual areas of the brain. It would be naive however to believe that the concept of “space” *i.e.* the stage where the geometrical shapes develop, is a straightforward one.

In fact as we shall see below this concept of “space” is still undergoing a drastic evolution.

The Cartesian frame allows one to encode a point of the Euclidean plane (or space) by two (or three) real numbers  $x^{\mu} \in \mathbb{R}$ . This irruption of “numbers” in geometry appears at first as an act of violence undergone by geometry thought of as a synthetic mental construct.

This “act of violence” inaugurates the duality between geometry and algebra, between the eye of the geometer and the computations of the algebraists, which run in time contrasting with the immediate perception of the visual intuition.

Far from being a sterile opposition, this duality becomes extremely fecund when geometry and algebra become allies to explore unknown lands as in the new algebraic geometry of the second half of the twentieth century or as in noncommutative geometry, two existing frontiers for the notion of space.

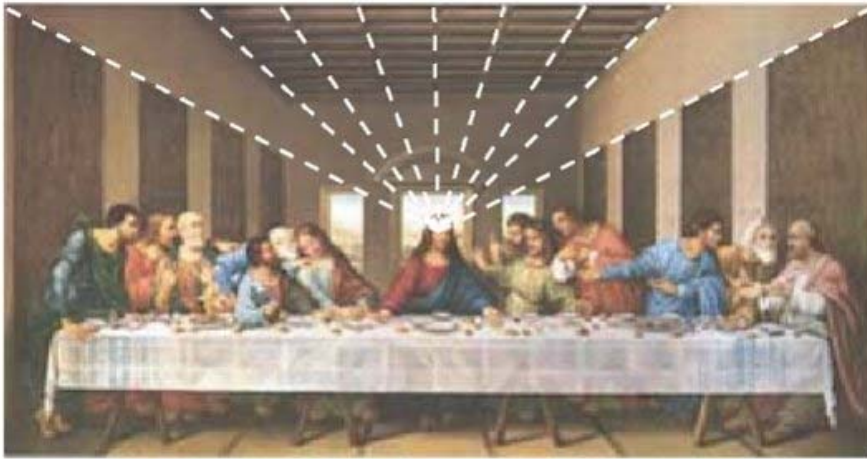


Figure 1: Perspective

### 2.1. Projective Geometry

Let us first briefly describe projective geometry a telling example of the above duality between geometry and algebra. In the middle of the XVIIIth century, G. Desargues, trying to give a mathematical foundation to the methods of perspective used by painters and architects founded real projective geometry. The real projective plane of Desargues is the set  $P_2(\mathbb{R})$  of lines through the origin in three space  $\mathbb{R}^3$ . This adds to the usual points of the plane a “line at infinity” which gives a perfect formulation and support for the empirical techniques of perspective.

In fact Desargues’s theorem (Figure 2) can be viewed as the base for the axiomatization of projective geometry.

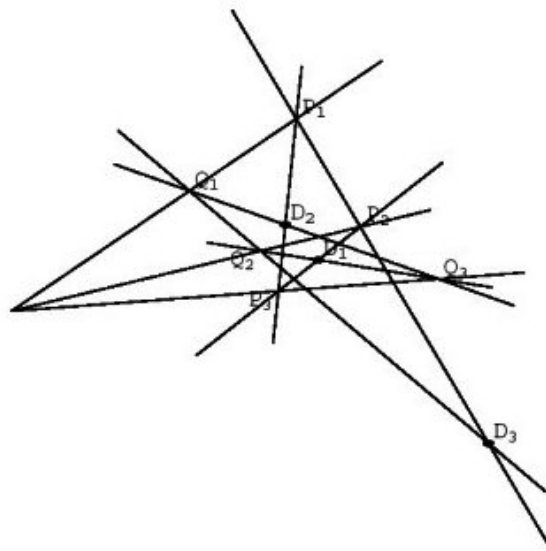


Figure 2: Desargues’s Theorem : *Let  $P_j$  and  $Q_j$ ,  $j \in \{1,2,3\}$  be points such that the three lines  $(P_j, Q_j)$  meet. Then the three points  $D_j := (P_k, P_l) \cap (Q_k, Q_l)$  are on the same line.*

This theorem is a consequence of the four extremely simple axioms which define projective geometry, but it requires for its proof that the dimension of the geometry be strictly larger than two.

These axioms express the properties of the relation " $P \in L$ " *i.e.* the point  $P$  belongs to the line  $L$ , they are:

- Existence and uniqueness of the straight line containing two distinct points.
- Two lines defined by four points located on two meeting lines actually meet in one point.
- Every line contains at least three points.
- There exists a finite set of points that generate the whole geometry by iterating the operation passing from two points to all points of the line they span.

In dimension  $n = 2$ , Desargues's theorem is no longer a consequence of the above axioms and one has to add it to the above four axioms. The Desarguan geometries of dimension  $n$  are exactly the projective spaces  $P_n(K)$  of a (not necessarily commutative) field  $K$ .

They are in this way in perfect duality with the key concept of algebra— that of field.

What is a field? It is a set of "numbers" that one can add, multiply and in which any non-zero element has an inverse so that all familiar rules are valid (except possibly the commutativity  $xy = yx$  of the product).

One basic example is given by the field  $\mathbb{Q}$  of rational numbers but there are many others such as the field  $\mathbb{F}_2$  with two elements or the field  $\mathbb{C}$  of complex numbers. The field  $\mathbb{H}$  of quaternions of Hamilton is a beautiful example of non-commutative field.

Complex projective geometry *i.e.* that of  $P_n(\mathbb{C})$  took its definitive form in "La Géométrie" of Monge in 1795. The presence of complex points on the side of the real ones simplifies considerably the overall picture and gives a rare harmony to the general theory by the simplicity and generality of the results.

For instance all circles of the plane pass through the "cyclic points" a pair of points (introduced by Poncelet) located on the line at infinity and having complex coordinates. Thus as two arbitrary conics any pair of circles actually meet in four points, a statement clearly false in the real plane.

The need for introducing and using complex numbers even to settle problems whose formulation is purely "real" had already appeared in the XVIth century for the resolution of the third degree equation.

Indeed even when the three roots of such an equation are real the conceptual form of these roots in terms of radicals necessarily passes through complex numbers. (*cf.* Chapters 11 to 23 in Cardano's book of 1545 *Ars magna sive de regulis algebraicis*).

## 2.2. The Angel of Geometry and the Devil of Algebra

The duality

$$\underline{\text{Geometry}} \quad | \quad \underline{\text{Algebra}} \tag{1}$$

already present in the above discussion of projective geometry allows us, when it is viewed as a mutual enhancement, to translate back and forth from geometry to algebra and obtain statements that would be hard to guess if one would stay confined to one of the two domains. This is best illustrated by a very simple example.

The geometric result, due to Frank Morley, deals with planar geometry and is one of the few results about the geometry of triangles that was apparently unknown to Greek mathematicians. You start with an arbitrary triangle  $ABC$  and trisect each angle, then you consider the intersection of consecutive trisectors, and obtain another triangle  $\alpha\beta\gamma$  (Fig.3). Now Morley’s theorem, which he found around 1899, says that *whichever triangle  $ABC$  you start from, the triangle  $\alpha\beta\gamma$  is always equilateral.*

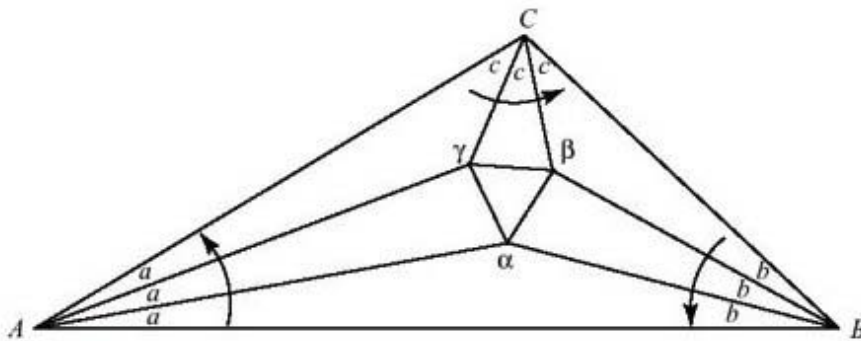


Figure 3: Morley’s Theorem : *The triangle  $\alpha\beta\gamma$  obtained from the intersection of consecutive trisectors of an arbitrary triangle  $ABC$  is always equilateral.*

Here now is an algebraic “transcription” of this result. We start with an arbitrary commutative field  $K$  and take three “affine” transformations of  $K$ . These are maps  $g$  from  $K$  to  $K$  of the form  $g(x) = \lambda x + \mu$ , where  $\lambda \neq 0$ . Given such a transformation the value of  $\lambda \in K$  is unique and noted  $\delta(g)$ . For  $g \in G$ ,  $g(x) = \lambda x + \mu$  not a translation, *i.e.*  $\lambda \neq 1$  one lets  $\text{fix}(g) = \alpha$  be the unique fixed point  $g(\alpha) = \alpha$  of  $g$ . These maps form a group  $G(K)$  (*cf.* Section 2.4) called the “affine group” and the algebraic counterpart of Morley’s theorem reads as follows

*Let  $f, g, h \in G$  be such that  $fg, gh, hf$  and  $fgh$  are not translations and let  $j = \delta(fgh)$ . The following two conditions are equivalent,*

- a)  $f^3 g^3 h^3 = 1$ .
- b)  $j^3 = 1$  and  $\alpha + j\beta + j^2\gamma = 0$  where  $\alpha = \text{fix}(fg)$ ,  $\beta = \text{fix}(gh)$ ,  $\gamma = \text{fix}(hf)$ .

This is a sufficiently general statement now, involving an arbitrary field  $K$  and its proof is a simple “verification”, which is a good test of the elementary skills in “algebra”.

It remains to show how it implies Morley’s result. But the fundamental property of “flatness” of Euclidean geometry, namely

$$a + b + c = \pi \tag{2}$$

where  $a, b, c$  are the angles of a triangle  $(A, B, C)$  is best captured algebraically by the equality

$$F G H = 1$$

in the affine group  $G(\mathbb{C})$  of the field  $K = \mathbb{C}$  of complex numbers, where  $F$  is the rotation of center  $A$  and angle  $2a$  and similarly for  $G$  and  $H$ . Thus if we let  $f$  be the rotation of center  $A$  and angle  $2a/3$  and similarly for  $g$  and  $h$  we get the condition  $f^3 g^3 h^3 = 1$ .

The above equivalence thus shows that  $\alpha + j\beta + j^2\gamma = 0$ , where  $\alpha, \beta, \gamma$ , are the fixed points of  $fg, gh$  et  $hf$  and where  $j = \delta(fgh)$  is a non-trivial cubic root of unity. The relation  $\alpha + j\beta + j^2\gamma = 0$  is a well-known characterization of equilateral triangles (it means  $\frac{\alpha-\beta}{\gamma-\beta} = -j^2$ , so that one passes from the vector  $\overrightarrow{\beta\gamma}$  to  $\overrightarrow{\beta\alpha}$  by a rotation of angle  $\pi/3$ ).

Finally it is easy to check that the fixed point  $\alpha, f(g(\alpha)) = \alpha$  is the intersection of the trisectors from  $A$  and  $B$  closest to the side  $AB$ . Indeed the rotation  $g$  moves it to its symmetric relative to  $AB$ , and  $f$  puts it back in place.

Thus we proved that the triangle  $(\alpha, \beta, \gamma)$  is equilateral. In fact we also get for free 18 equilateral triangles obtained by picking other solutions of  $f^3 = F$  etc...

This is typical of the power of the duality between on the one hand the visual perception (where the geometrical facts can be sort of obvious) and on the other hand the algebraic understanding.

Then, provided one can write things in algebraic terms, one enhances their power and makes them applicable in totally different circumstances. For instance the above theorem holds for a finite field, it holds for instance for the field  $F_4$  which has cubic roots of unity.... So somehow, passing from the geometrical intuition to the algebraic formulation allows one to increase the power of the original “obvious” fact, a bit like language can enhance the strength of perception, in using the “right words”.



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