MATHEMATICS THROUGH MILLENIA

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Contents

Summary

This chapter provides a brief tour through the history of mathematics from the very beginnings to modern times, with an emphasis on the main contributions and important periods of mathematics in various civilizations.

1. Introduction

The origin of mathematics stretches far back in time. Exactly how far is difficult to decide, but there are known artifacts with indication of mathematical activities from many thousands of years ago. Mathematics developed unevenly through several millennia with relatively short periods of time producing new discoveries, followed by long periods with little mathematical activity. Since the scientific revolution in the sixteenth and seventeenth centuries, mathematics has experienced a tremendous growth and is now recognized as one of the basic underpinnings of modern civilization.

Writing a short history of mathematics involves many choices. The cultural background of the author and the author's overall knowledge of mathematics in its broadest sense will undoubtedly bias these choices - different authors might have focused on other aspects. The author hopes nevertheless that his exposition has found an acceptable balance.

2. The Dawn of Mathematics

Mathematics originated in the early needs of mankind for counting and measuring in relation to both quantities and spatial objects. An interesting early mathematical artifact is the *Ishango bone* found in 1960 on the shores of Lake Edward on the border of Uganda and Zaire. The bone is named after a small settlement living at this location in prehistoric times and carbon-dated to be about twenty thousand years old. The Ishango man apparently carved the bone according to some kind of pattern, but it is not clear what the strokes or notches on the bone exactly represent. The carvings are most likely just a representation of numbers but maybe even some arithmetic was done. It has also been suggested that the bone could have been a lunar calendar, but it all remains speculations. After the Ishango bone was found in 1960, it seems likely that mathematics in ancient Egypt in relation to the pyramids and surveying have an African background.

2.1. Egyptian Mathematics

The development of a civilization in old Egypt was strongly dependent on the river Nile with its tides and annual flooding. The flooding was decisive for farming and life in Egypt and hence keeping track of land areas by surveying was an important subject. From a mathematical point of view, it is interesting that the name of the mathematical field *geometry* derives from the Greek word *geometria* meaning the measurement of land, in other words surveying. The Greek historian Herodotus used this word in the fifth century BC in his great epic on the Persian wars in which he wrote that 'geometria' was used in old Egypt to find the right distribution of land after the flooding of the Nile.

The main achievements of early Egyptian mathematicians involved the practical skill of measurement not least in relation to the building of the magnificent pyramids, the oldest dating from about 2700 BC. In particular, the Great Pyramid of Cheops has a square base whose sides of length 230-metres agree to less than 0.01 %.

Our main source of ancient Egyptian mathematics is a papyrus from about 1650 BC, copied from an older text by Ahmes no longer in existence from about 1850 BC. The papyrus is named the *Rhind papyrus* after the British explorer Henry Rhind who bought it in 1858; it is now in the British Museum. The papyrus contains tables of numbers and a collection of about 80 problems used for the education of scribes. Some of the problems are formulated in a practical setting such as to determine how much bread or how much beer that can be produced from a given amount of grain, to compute the area of a rectangular field, or the volume of a cylindrical silo, or to determine the slope of a pyramid. Many problems are, however, not related to practical situations. The solutions were found by rules the scribes were taught - many of the problems include partial or full solutions. One rule is about the area of a circle and says that you find the area by taking the diameter, subtracting 1/9 of it and squaring the remaining part. This procedure gives a fairly good approximation to the exact area of a circle and corresponds to an approximate value of π about 3.16 in the current notation.

The Egyptian notion of numbers consisted of the positive integers, unit fractions (reciprocals of integers) and the fraction 2/3. They had symbols for the numbers 1, 10, 100, etc., up to 1000000 and constructed symbols for other positive integers by repeating these symbols. They had elaborate procedures for multiplications and divisions based on a simple method - doubling and halving.

2.2. Mesopotamian Mathematics

Another ancient civilization developed in Mesopotamia, between the rivers Euphrates and Tigris in present-day Iraq sometime around 3500 BC. The Sumerians were the first people to build larger cities. Around 1700 BC, the ruler of the city Babylon conquered the entire area around his city and it developed into an impressive metropolis. Later the name Babylonia slightly incorrectly has been attached to all cultural and scientific achievements of the people in Mesopotamia.

Mesopotamian, or Babylonian, mathematics is the best examined mathematical tradition before the Greeks and probably the one with the largest impact on later developments in mathematics. The main reason for this is that in Mesopotamia writing was done by means of a stylus on clay tablets, which, unlike papyrus, will survive. The beginnings of mathematics in Mesopotamian can be traced back to around 3300 BC, when systematic bookkeeping and measurement of land areas caused developments in measurement systems.

Around 2500 BC it became a specialized occupation for a select few in Babylonia to do writing and calculations. For use in the strongly centralized and bureaucratic administration, the first position-based number system was developed shortly before 2000 BC. The system was based on the number 60 (sexagecimal system), and elements of the system can still be found in present day division of time and angle. As inevitable with real mathematicians, the scribes extended, however, their mathematical ideas far beyond the limits of practical necessity. For training purposes they created the first purely mathematical exercises, for example division with large round numbers including numbers for which the reciprocal is a non-terminating sexagecimal fraction.

Babylonian mathematics reached its peak around 1800 BC, where methods for solving equations of first and second degree were developed, based on algebraic techniques such as completing the square – used to present day. Only the solution method for linear equations were used for training of students, whereas the more refined exercises on the excavated source material in the form of clay tablets, served the purposes of training skills of computation and to demonstrate technical mastery among the teachers.

After 1600 BC, the schools for professional writers dissolved and the work with more advanced mathematics came to a halt. The latter half of the first millennium BC saw a new blossoming in Babylonian mathematics in connection with refined computations in astronomy.

2.3. Mayan Mathematics

The Mayans are the creators of the most advanced civilization in ancient America. It lasted for over 3000 years, from around 2000 BC to 1521 when Spanish conquerors invaded Mexico. The Spaniards discovered that the Mayans had produced a great empire controlled by a network of city-states with dense populations. They had a system of written language in the form of intricate hieroglyphs written on both stone and paper that was made out of bark. The Mayans were avid astronomers, and watched the stars closely, and they developed a very accurate calendar based on the movements of the sun, the moon, and Venus. They had very developed arts such as sculpture, painting and pottery, and sciences such as medicine. There were many traders in their society, and many others were farmers. They grew corn and other crops in an advanced agricultural system that included irrigation.

The Mayans had developed an interesting number system with twenty symbols representing the numbers one to twenty. They wrote out numbers using bars and dots, where each dot represents 1 and each bar represents 5. Placing dots on top of bars constructed the symbols for 1 to 20. Base 20 systems are still in use today by such tribes as the Hopi and the Inuits. Almost certainly the reason for base twenty arose from ancient people who counted on both their fingers and their toes. The total number system of the Mayans was based on 18 and 20 so as to include 360.

The Mayans carried out astronomical measurements with remarkable accuracy, using two sticks in the form of a cross and viewing astronomical objects through the right angle formed by the sticks. With such crude instruments the Mayans were able to calculate the length of the year to be 365.242 days (the modern value is 365.242198 days). Two further remarkable calculations are of the length of the lunar month. At Copán (now on the border between Honduras and Guatemala) the Mayan astronomers found that 149 lunar months lasted 4400 days. This gives 29.5302 days as the length of the lunar month. At Palenque in Tabasco they calculated that 81 lunar months lasted 2392 days. This gives 29.5308 days as the length of the lunar month. The modern value is 29.53059 days.

3. The Greek Heritage in Mathematics

Around 600 BC, the Greeks seriously began to consider mathematics as a logical structure and as a tool for understanding cosmos. It is difficult to say why; all one can say for sure is that from about that time, the Greeks were convinced that in all essentials the universe is rationally organized, and that all the phenomena of nature pursue a precise and stable plan; indeed a mathematical plan. Thus the Greeks became the first people to attempt to reason their way to explanations of natural phenomena.

3.1. Geometry

The mathematical arguments of the Greeks were largely geometric, and while they explored things, they developed aids from geometry, so that others might more readily reach the frontiers and help to achieve new conquests in the understanding of the universe. In this way, the Greeks contributed to the founding of the methodology of modern science.

Geometry as pure mathematics encompasses a collection of abstract statements about ideal forms and proofs of these statements. Thales (about 600 BC) is the first Greek to be mentioned in the process towards building a hierarchical system for geometry based on initial axioms and deduction and reasoning. According to legend, Thales proved several theorems in geometry, among them theorems about circles and congruence of triangles, applying it to navigation at sea. The members of the famous school of Pythagoras about 500 BC believed that 'All is number' and tried to quantify everything paying particular attention to figurate numbers (e.g. triangular), and integers (e.g. prime numbers). In geometry the Pythagoreans are known for their discovery of the so-called *Pythagorean theorem,* that the area of the square on the hypotenuse of a right-angled triangle is the sum of the areas of the squares on the other two sides; it is debated among scholars whether the Babylonians knew the theorem already. From the early period one should, however, in particular single out Eudoxus (c. 391-338 BC), who is known for a *theory of proportions* and the so-called *method of exhaustion* making rigorous determinations of areas and volumes possible.

From about 500 to 300 BC, Athens was the most important intellectual center in Greece, having among its scholars Plato (c. 427-347 BC) and Aristotle (384-322 BC). Neither is remembered primarily as a mathematician, but they certainly prepared the way for the mathematical achievements of the Greeks to come. Around 387 BC, Plato founded his school in a part of Athens called the Academy. The Academy soon became the focal point for mathematical study and philosophical research. Over the entrance appeared the inscription: "Let no-one ignorant of geometry enter here".

Plato believed that the study of mathematics and philosophy provided the finest training for those who were to hold positions of responsibility in the state. In his *Republic* he discussed the Pythagoreans' mathematical arts of arithmetic, plane and solid geometry, astronomy and music, explaining their nature and justifying their importance for the training of statesmen. His *Timaeus* includes a discussion of the five regular solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron. Aristotle was fascinated by logical questions and systematized the study of logic and deductive reasoning. In particular, he mentioned a proof that $\sqrt{2}$ cannot be written in rational form as a quotient of integers, and he discussed syllogisms.

Around 300 BC, with the rise to power of Ptolomy I, mathematical activity moved to the Egyptian part of the Greek empire. In the metropolis Alexandria, Ptolomy founded a university that became the intellectual center for Greek scholarship for over 800 years.

Classical Greek geometry has first of all survived (though not in primary source material) through the famous 13 books written by Euclid in Alexandria around 300 BC and known as Euclid's *Elements.* In these books the mathematical (in particular, the geometrical) knowledge possessed by the Greeks at the time of Euclid is summarized and systematized in such a way that the exposition has put a stamp on mathematical writings ever since. Eudoxus is often credited with developing the theory behind Books V (on proportion) and XII (on the method of exhaustion). The geometrical content in the *Elements* is now known as *Euclidean geometry*.

Euclid based his studies of geometry on five initial axioms and postulates and used rules of deduction to derive each new proposition in a logical and systematic order. The first three postulates establish the admissible rules for constructions with ruler and compass. Such constructions were of great importance to the Greeks, since they provided existence proofs for geometrical objects. The fourth postulate introduces the notion of a right angle and postulates its uniqueness. The first four postulates were accepted right away. The fifth postulate caused more concern and is the most famous. It is formulated as a postulate about intersecting straight lines, which to Euclid means line segments of finite extension - the Greeks avoided the infinite that they did not grasp. In the following centuries up to the beginnings of the nineteenth century many attempts were made in vain to prove that Euclid's fifth postulate followed from the other four postulates. During this work, the postulate was reformulated in many equivalent ways and finally by Playfair in 1795 into what is now known as *the parallel postulate*.

The method of exhaustion, developed by Eudoxus, was used to approximate a geometrical figure for which the area (volume) had to be determined by geometrical figures for which the area (volume) was already known. When the approximation is made better and better, the area (volume) searched for is obtained as a limit value. The Greeks, though, did not accept infinite processes; in particular they had troubles with formalizing the notion of a limit. The work of Eudoxus was refined in writings of Archimedes (c. 287-212 BC) born and later working in the Greek settlement Syracuse in Sicily after having received his education in Alexandria. Archimedes was one of the greatest mathematicians of all times. He perfected the method of exhaustion and obtained eminent results such as determining the area of the surface of a sphere. He also listed the thirteen semi-regular solids whose faces are regular polygons but not all of the same shape. By considering 96-sided polygons that approximate a circle, Archimedes proved that the number π lies between $3\ ^{10}/_{71}$ and $3\ ^{10}/_{70}$ (the familiar approximation

$^{22}/_{7}$).

Apollonius (c. 262-190 BC) was another great figure in Greek mathematics. He was born in Perga in the northwestern part of Asia Minor, but came to Alexandria in his youth and learned mathematics from Euclid's successors. Apollonius is primarily known for his systematic treatment of the *conic sections:* the ellipse, parabola and hyperbola.

3.2. Number Theory

Although the ancient Greeks admired geometry and made their largest discoveries in this field, they also contributed to other mathematical areas, in particular to number theory. Famous is the proof in Euclid's *Elements* that there are infinitely many prime numbers, using *reductio ad absurdum* (proof by contradiction), a favorite type of proof for Euclid. Another well-known result in the *Elements* is the division algorithm to find the greatest common divisor of two integers.

Diophantus of Alexandria obtained other remarkable results in number theory. The dates of Diophantus are debatable but it seems generally accepted that he lived around 250 AD. In his *Aritmetica* he was primarily occupied by finding integer solutions to algebraic equations with integer coefficients, nowadays called *Diophantine equations*.

The pioneering contributions by the Greek mathematicians exhausted the possibilities of elementary mathematics to the extent that almost no significant progress was made, beyond what is called Greek mathematics, until the sixteenth century. What is really amazing is that Greek immigrants in Alexandria, and on Sicily did all of the really significant work in the relatively short interval from about 350 BC to 200 AD.

4. The Golden Period of the Hindus and the Arabs in Mathematics

The mental successors of the Greeks in the history of mathematics were the Hindus of India. Though the Hindus had already made ingenious developments in mathematics much earlier, the contributions to mathematics from this culture became really significant only after being merged with the Greek achievements by Islamic scholars.

4.1. Hindu Mathematics

The Hindu civilization dates back to at least 2000 BC, and mathematical traditions can be followed back to about 600 BC. Among the religious writings from this early period was a class called Śulvasūtras (rules of the cord), containing instructions for construction of altars. In designing the permissible altars the Hindus developed good approximations to $\sqrt{2}$ and found solutions to problems corresponding to quadratic equations – actually the same solutions as found by Babylonian mathematicians much earlier. In geometry they probably knew rules corresponding to the Pythagorean theorem. The mathematical rules stated were not followed by proofs, but were entirely empirical.

From about 300 BC mathematics was important in the Jain religion as well for practical calculations in astronomy as for speculations about very large numbers. In this period the Hindus borrowed elements of Babylonian, and later also Greek, mathematical astronomy. The geometry of the Hindus was certainly Greek, but they did have a special gift for arithmetic, where they went very far on their own. As to algebra, they may have borrowed from Alexandria and possibly directly from Babylonia, and India was also somewhat indebted to China.

The real blossoming of mathematics in India took place in the period from about 500 to 1200 with Aryabhata (476-550) and Brahmagupta (598-670) as the two most outstanding Indian mathematicians, and later Bhaskara (1114-c.1185), who published important works on arithmetic. Most work was motivated by astronomy and astrology. In fact, there were no separate mathematical texts, and the mathematical material was presented in chapters of works on astronomy. In the beginning of this period (at the latest), Hindu mathematicians created the modern position based number system, though not decimal fractions. Later they introduced the negative numbers to represent debts, and they started using zero in the way we now use this number, i.e. as a number to calculate with – not just a placeholder. Aryabhata gave the first systematic treatment of Diophantine equations. He also presented formulas for the sum of the numbers and of their squares and cubes and obtained the value 3.1416 for π. Brahmagupta discussed the use of zero and made the first known use of negative numbers about 628. He also solved some quadratic Diophantine equations such as $92x^2 + 1 = y^2$ (now known as *Pell's equation*), for which he obtained the integer solution x = 120, y =1151. In this period, the Hindus also developed the basis for modern trigonometry by conversion of Greek ideas.

After 1350 AD trigonometry was further developed in particular in Southern India by the introduction of series expansions, later to be found in Europe in connection with the methods from calculus, with a main contribution by Nilakantha in a work of 1501. Elements of combinatorics were developed in connection with the study of metre in poetry.

The Hindus were interested in, and contributed to, the arithmetical and computational activities of mathematics rather than to deductive patterns. Their name for mathematics was *ganita*, which means 'the science of calculation'.

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Biographical Sketch

Vagn Lundsgaard Hansen is born September 27, 1940 in Vejle, Denmark. He is professor of mathematics since 1980 at the Technical University of Denmark. He earned a master's degree in mathematics and physics from the University of Aarhus, Denmark, 1966, and a Ph.D. in mathematics from the University of Warwick, England, 1972. He has held positions as assistant professor, University of Aarhus, 1966-69; research fellow, University of Warwick, 1969-72; associate professor, University of Copenhagen, Denmark, 1972-80. He was visiting professor (fall 1986), University of Maryland, College Park, US.

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