# MATHEMATICS ALIVE AND IN ACTION

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**Keywords:** Mathematical work, pure and applied mathematics, mathematical modeling in the sciences, mathematics and the arts, mathematics in industry, impact of mathematics on society.

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**Summary**

This chapter addresses two main questions: What do mathematicians do? What is mathematics good for? With focus on recent times, a panorama of mathematical contributions to civilization is presented and the intellectual drive by which they were perceived is described.

## 1. Introduction

Mathematics is the science of structures and patterns. Mathematicians look for structures in nature, patterns in daily life, structures essential for the functioning of society, etc., seeking rational explanations of the phenomena, and building abstract models by which to predict and control them.

The drive for investigating the abstract pure structures in mathematics in isolation from the concrete origins of the structures varies among mathematicians. Most mathematicians will agree though that mathematics has the dual nature of combining abstract thinking with the description of concrete phenomena.

Abstract mathematics has become a driving force in many sciences and lies behind many deep and important developments in society. In addition to applications of vintage mathematics, probabilistic and stochastic ways of thinking, and even very recent mathematical results, are now an integrated part of high-tech constructions and decision-making of all kinds.

## 2. Fundamental Mathematical Research

Most people are surprised to learn that mathematical research exists as a science pursued by thousands of people all over the world. They have the impression that there is no mathematics beyond school mathematics: arithmetic, algebra, geometry, and maybe trigonometry and calculus. But even parts of school mathematics, for example, the formulation of the 'laws of algebra', the definition of the integral, the concept of function, and even fundamental questions about the number system were research questions at the beginning of the twentieth century.

Mathematics that seems fundamental today very often seemed strange when it first saw the light of day. As an example, *linear algebra*, including the abstract notion of *vector spaces* suggested by the Italian mathematician G. Peano (1858-1932) in 1888, was heavily criticized as a useless abstraction for the first several decades of its existence. Since then linear algebra has grown into a fundamental mathematical discipline with far reaching applications in the technical sciences (notably in optimization theory) and the social sciences (notably in econometrics). Nowadays, linear algebra belongs to the core curriculum of mathematics at university level worldwide, and one of its many applications, *linear programming*, is important for all sorts of planning, scheduling and distribution of goods. It is difficult to predict which of the many lines of mathematical investigations pursued today will prove invaluable, and which will deserve forgetting.

### 2.1. Drive of Mathematicians

For many mathematicians the drive in their work is related to solving mathematical problems. Some problems, such as the last theorem of Fermat, have a long history and hence give high prestige to the people who solve them. At the fundamental level of mathematics, the problems are not chosen to satisfy an urgent need prompted by science or society. Most problems arise as part of ongoing research in a larger field and the depth of the problems, and the potential applications of the theory in which the solutions are embodied, may not be immediate. On the other hand, history shows surprisingly many examples that a mathematical theory has had important applications after it was developed. A well-known example is Maxwell's laws of electromagnetism, which are mathematical laws with huge implications for society. Another example is the mathematical laws of aerodynamics, which were discovered by scientists thinking about the basic laws of nature and not with any specific application in mind.

## 2.2 Pure Mathematics at the Turn of Millennium

From around 1970, the study of the mathematical properties of knotting and braiding – the study of how strings wind around themselves in space – experienced a new blossoming. That these phenomena are closely related was formally established already in 1923, when the American mathematician J.W. Alexander (1888-1971) showed that every knot, or more generally a link, arises by closing the ends of a braid. The theory of braids, knots and links got real momentum around 1985 when the New Zealand mathematician Vaughan Jones discovered a new strong polynomial invariant for knots which could distinguish between the types of many more knots than before. Jones discovered his polynomial by studying a representation of the braid group as a group of operators in certain *operator algebras* (von Neumann algebras) and taking the traces of the operators. The theory of braids, knots and links is one of the most promising topics in mathematics today and seems to hold the secret to a number of problems in the sciences. In fact, braids may be the success to solving many problems of physics.

Protein folding is very closely related to this process. As they gain deeper and deeper understanding and large amounts of data become available, biologists find that more mathematical theory is needed. At the moment there are few people who know both mathematics and biology at a sufficiently high level, but the number is growing rapidly. It seems probable that a deep understanding of genetics is dependent on understanding knotting.

Algebra and number theory are other areas of mathematical research currently with a high level of activity. Abstract algebra in general got new momentum after Andrew Wiles' proof of the famous Fermat theorem in 1994. Research in number theory has also been enhanced by its applications in *coding theory* and *cryptography*. For example, arithmetic over prime numbers has been used for the generation of optimal codes. From algebra we mention that infinite-dimensional representations of groups have suggested a new design of large, economically efficient networks of high connectivity.

In mathematical analysis, the theory of *wavelets* is an area of research with a high level of activity. The basic idea of wavelets arose quite naturally in electrical engineering in connection with analysis of signals, where wavelets offer more flexible and sparse approximations to functions in terms of infinite series than classical Fourier theory. As a mathematical subject, the theory of wavelets crystallized in the 1980s and has already by now a great number of industrial applications.

## 3. Theoretical Computer Science

Theoretical *computer science* is maybe one of the most important and active areas of scientific study today. As a discipline, theoretical computer science was founded in the 1930s, before electronic computers existed, in works of the English mathematician Alan Turing (1912-1954) and his contemporaries who set out to mathematically define the concept of 'computation' and to study its power and limits. During the Second World

War, the first electronic computers were built in England and the United States for deciphering German military codes. Inspired by these computers, the Hungarian mathematician John von Neumann (1903-1957) contributed to the development of a 'stored program computer' in which the data and instructions are held in an internal store until needed.

The practical use of computers, and the unexpected depth of the concept of 'computation', has significantly expanded theoretical computer science, and it has grown into an exciting field with many connections to other sciences. During its evolution, the focus of the field has changed from the notion of 'computation' to the more elusive concept of 'efficient computation'.

### 3.1 Computation in Polynomial Time

The question of 'efficient computation' is related to the philosophical question of what is knowable and what is unknowable. In the 1930s, the Austrian logician Kurt Gödel (1906-1978) established that complete certainty could not be found in arithmetic – the *Gödel incompleteness theorem*. In the same period, rules to decide what is computable and what is not computable were formulated by Alan Turing. This led to the more refined question about what is computable in polynomial time, or short, P-time. If the computer time needed for a problem is not polynomial, the problem becomes computationally intractable for growing parameters. The level of computational intractability may, however, possibly vary from problem to problem. A problem whose solution time is bounded by a single P-time verification and hence is computationally tractable is said to belong to the class P. A problem for which a postulated solution (to any instance of the problem) can be checked in P-time (relative to the size of the instance) is said to belong to the class NP. Here NP stands for Non-deterministic Polynomial. As an example, consider the problem of writing a large natural number $n$ as a product of smaller factors – *the factorization problem*. This is an NP-problem. For example, suppose a witness claims that $n = p \cdot q$. Then we can check in P-time if this is true or false because the product $p \cdot q$ can be computed in P-time.

For any given computational problem, which is known to be an NP-problem, the question is to decide whether it is actually a P-problem. This question is known as one of 'P versus NP'. A famous open question of 'P versus NP' in a problem concerns *the traveling salesman* who needs to visit a number of cities minimizing the length of his route**.** A computer program can easily be written to compute a shortest possible route, but as the number of cities becomes larger, computer time may increase more than exponentially, and it is not known whether the *traveling salesman problem* is a P-problem. In fact, a positive solution to this question would imply that NP = P. Most cryptographic codes in use today are based on factorization of large numbers and they are designed on the assumption that computing the factors is not known to be a P-problem. This assumption has, of course, enormous implications for safe use of electronically stored information, where large numbers are used as security codes.

There are some very interesting current developments on the 'P versus NP' question, which may be related to the Gödel incompleteness theorem. It seems possible that certain mathematical statements that eventually include lower bounds on computation, such as 'P does not equal NP', cannot be proved within the framework of arithmetic building on Peano's axioms, or set theory.

The emergence of the complexity-based notion of one-way functions in theoretical computer science, together with the use of randomness, led to the development of modern *cryptography*. More generally, *complexity theory*, which attempts to classify problems according to their computational difficulty, has integrated many of these ideas and has given rise to the field of *proof complexity*, where the goal is to quantify what constitutes a difficult proof.

Some of the fundamental problems in theoretical computer science have gained prominence as central problems of mathematics in general, notably 'P versus NP', and the computational aspects play an increasingly significant role in all kinds of mathematical research. The very notion of 'computation', and the major problems surrounding it, have taken on deep philosophical meaning and consequences. In addition to the 'P versus NP' question, the field is focused on a few clear and deep questions, like for example, whether randomization does help computation, about what constitutes a difficult theorem to prove, and whether quantum mechanics can be effectively simulated by classical means.

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**Biographical Sketch**

**Vagn Lundsgaard Hansen** is born September 27, 1940 in Vejle, Denmark. He is professor of mathematics since 1980 at the Technical University of Denmark. He earned a master's degree in mathematics and physics from the University of Aarhus, Denmark, 1966, and a Ph.D. in mathematics from the University of Warwick, England, 1972. He has held positions as assistant professor, University of Aarhus, 1966-69; research fellow, University of Warwick, 1969-72; associate professor, University of Copenhagen, Denmark, 1972-80. He was visiting professor (fall 1986), University of Maryland, College Park, US.

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He is Chairman Committee for Raising Public Awareness of Mathematics appointed by the European Mathematical Society, 2000-2006. He was Invited speaker International Congress of Mathematicians, Beijing 2002 and Invited regular lecturer 10$^{th}$ International Congress on Mathematical Education, Copenhagen 2004. He is President Danish Academy of Natural Sciences since 1984, Member European Academy of Sciences (Brussels) 2004, Member Danish Natural Science Research Council, 1992-98, and functioned for four years in this period as vice-chairman.