RINGS AND MODULES

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Contents

Summary

Rings are sets with two algebraic operations called addition and multiplication. Their basic properties are explained with many typical examples. Then modules over rings are introduced as generalizations of vector spaces.

Rings and modules are not only indispensable in dealing with fields and algebraic equations in *Fields and Algebraic Equations* but also play crucial roles in number theory and algebraic geometry.

Results on matrices and linear algebra in *Matrices, Vectors, Determinants and Linear Algebra* and on groups in *Groups and Applications* will be freely used.

1. Definition of Rings

A *ring* is a set R endowed with the *addition* $R \times R \ni (x, y) \mapsto x + y \in R$ and the *multiplication* $R \times R \ni (x, y) \mapsto xy \in R$ satisfying the following properties:

- (Additive group) With respect to the addition, R is a commutative group with $0 \in R$, and $-x \in R$ for each $x \in R$ satisfying $x + (-x) = 0$.
- (Distributivity) $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$ hold for any $x, y, z \in R$.

The main concern here will be rings that are *associative* and *commutative* with the *unity*, that is,

- (Associativity) $x(yz) = (xy)z$ holds for any $x, y, z \in R$.

- (Commutativity) $xy = yx$ holds for any $x, y \in R$.
- (Existence of the unity) There exists $1 \in R$ such that $1x = x1 = x$ holds for any $x \in R$.

However, here are important examples that do *not* satisfy these properties.

Lie algebras: In this case, the multiplication is usually denoted by the "Lie bracket" $R \times R \ni (x, y) \mapsto [x, y] \in R$. It is required to satisfy:

- $[x, x] = 0$ for any $x \in R$. In view of the distributivity, one thus has $[x, y] = -[y, x]$ for any $x, y \in R$.
- Instead of the associativity, *Jacobi's identity*
  $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$
  holds for any $x, y, z \in R$.

Here are typical examples:

- The set $R := M_n(\mathbb{R})$ of $n \times n$ real matrices with the usual matrix addition $\mathbf{A} + \mathbf{B}$ and the Lie bracket $[\mathbf{A}, \mathbf{B}] := \mathbf{AB} - \mathbf{BA}$ for $\mathbf{A}, \mathbf{B} \in M_n(\mathbb{R})$ is a Lie algebra, where $\mathbf{AB}$ and $\mathbf{BA}$ are the usual matrix multiplication.
- Let $\mathbb{Q}[t] := \mathbb{Q}[t_1, t_2, \ldots, t_n]$ be the set of polynomials in n variables $t_1, t_2, \ldots, t_n$ with rational coefficients. A map $D : \mathbb{Q}[t] \to \mathbb{Q}[t]$ is called a *derivation* if $D(f + g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$ hold for any $f, g \in \mathbb{Q}[t]$. The set $\mathrm{Der}(\mathbb{Q}[t])$ of derivations of $\mathbb{Q}[t]$ is a Lie algebra under the addition $D + D'$ defined by $(D + D')(f) := D(f) + D'(f)$ for any $f \in \mathbb{Q}[t]$ and the Lie bracket $[D, D'] := D \circ D' - D' \circ D$, where the circle "$\circ$" means the composition of maps. In this example, any D can be written as a "linear combination with polynomial coefficients"

$$D = f_1(t)\frac{\partial}{\partial t_1} + f_2(t)\frac{\partial}{\partial t_2} + \cdots + f_n(t)\frac{\partial}{\partial t_n}$$

  of the usual partial derivatives $\partial/\partial t_1, \ldots, \partial/\partial t_n$. Lie algebras are used, for instance, for "infinitesimal" study of Lie groups, which play instrumental roles in geometry.

Non-commutative associative rings: Just as commutative rings play major roles in algebraic geometry as will be seen later, non-commutative associative rings of "operators" play important roles in "non-commutative geometry". Here are elementary examples:

- For $n \geq 2$, the set $M_n(\mathbb{R})$ of $n \times n$ real matrices is a non-commutative associative ring under the usual matrix addition $\mathbf{A} + \mathbf{B}$ and matrix multiplication $\mathbf{AB}$ for $\mathbf{A}, \mathbf{B} \in M_n(\mathbb{R})$.
- The set $\mathbb{H} := \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ of *Hamilton's quaternions* is a non-commutative

associative ring under the addition and multiplication of quaternions.

- The set $M_n(\mathbb{H})$ of $n \times n$ matrices with Hamilton's quaternion as entries is a non-commutative associative ring under the matrix addition and matrix multiplication, which makes sense even though the multiplication of entries is not commutative.

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Biographical Sketch

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