BASIC NOTIONS OF GEOMETRY AND EUCLIDEAN GEOMETRY

Tetsuya Ozawa
Department of Mathematics, Meijo University, Japan

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Summary

After preparing the notation and terminology on metric spaces and transformation groups, briefly discussed are basic concepts about a positive definite inner product on real vector space. Then Euclidean space is introduced by defining a natural distance function (called the Euclidean distance) on a real vector space. On plane geometry of triangles, several formulae relating with elements of a general triangle and associated circles are shown. Algebraic and geometric characterizations of conic sections including

the directrix and eccentricity are explained, and then it is shown that the trajectory of a motion under the influence of inverse square central force is a conic section. In relation to Platonic solids as well as prism and pyramid, the enumerations of finite subgroups and a certain type of discrete subgroups which concern crystallography in two and three-dimensional Euclidean spaces are discussed.

1. Introduction

The first systematic treatment of geometry based on the flat plane and space was established by Greek mathematician *Euclid* in his text *Elements*. The flat and indefinitely continued space modeled after what one ordinarily experiences is called the *Euclidean space*, which was recognized to be endowed with a canonical distance function. Transformations that preserve the distance are freely used to establish congruences between two figures in plane and space.

The simplest way to construct a Euclidean space is to introduce a certain distance function on the real coordinate space, which is regarded as a real vector space and carries a natural inner product, called the dot product, where the distance is defined by using the Pythagorean theorem of a right triangle. By the help of linear algebra, one expresses the set of isometries as a semi-direct product of the orthogonal group and the linear space.

Trigonometric functions simply relate the inner angles to the ratios of the lengths of edges of a right triangle. Moreover, through the law of sines and cosines, they provide various formulae related to the elements of a general triangle, and also to the radii of circles, such as the circumscribed circle, the inscribed and escribed circles of the triangle.

The study of conic section is a subject of Euclidean geometry, which played an important role in the history of natural science, that is, in a revolutionary step of the discovery of Newton’s theory of universal gravitation and mechanics. One of the most famous achievement in Newton’s discovery is to find that the trajectory of a point mass moving in the influence of inverse square central force is a conic section.

In Euclidean geometry, repeating patterns such as regular tessellations are studied with their symmetry groups, that is, the set of isometries under which the images of the pattern remain unchanged. One important application of such a study is the classification of crystal structures of atoms and molecules. On the other hand, the Platonic solids have been stimulating people for long time. Their symmetry groups are popular not only in geometry but also in many branches of mathematics. Investigating those groups is a step to enumerate the symmetry groups of crystal structures.

2. Basic Notions

2.1. Metric Space

A point set M is called a metric space, if it is endowed with a real valued function $d : M \times M \to \mathbf{R}$ that satisfies

1. $d(p,q) \geq 0$ for any $p,q \in M$, and $d(p,q) = 0$ if and only if $p = q$, (*Positivity*)
2. $d(p,q) = d(q,p)$ for any $p,q \in M$ (*Symmetricity*)
3. $d(p,q) + d(q,r) \geq d(p,r)$ for any $p,q,r \in M$ (*Triangle inequality*)

The function d is called the *distance function*, and the value $d(p,q)$ the *distance* between p and q. In order to specify the distance function d on M, one denotes the metric space $\mathcal{M} = (M,d)$.

An *isometry* from a metric space $\mathcal{M}_1 = (M_1, d_1)$ to another $\mathcal{M}_2 = (M_2, d_2)$ is, by definition, a map $\varphi : M_1 \to M_2$ such that, for any $p,q \in M_1$,

$$d_2(\varphi(p), \varphi(q)) = d_1(p,q).$$

An isometry is necessarily injective. If an isometry is surjective, then the inverse is also an isometry. The composition of two isometries is an isometry. For two metric spaces $\mathcal{M}$, $\mathcal{N}$, if there exists a bijective isometry between them, then $\mathcal{M}$ and $\mathcal{N}$ are said to be *isometric*. The set of all isometries from $\mathcal{M}$ to $\mathcal{N}$ will be denoted by Isom $(\mathcal{M}, \mathcal{N})$, and the set of all bijective isometries from $\mathcal{M}$ to $\mathcal{M}$ will be denoted by Isom$(\mathcal{M})$. Isom $(\mathcal{M})$ is regarded as a group, where the group multiplication is the composition of isometries, and the inverse element is the inverse map.

The topology of a metric space (M,d) is naturally introduced by using the fundamental system of neighborhoods $\{U_\epsilon(p); \epsilon > 0, p \in M\}$ defined by

$$U_\epsilon(p) = \{q \in M; d(p,q) < \epsilon\}.$$

2.2. Transformation Group

Let G be a group with the unit element e, and X a set. If a map $\varphi : G \times X \to X$ satisfies

1. $\varphi(e,x) = x$ for any $x \in X$,
2. $\varphi(g, \varphi(h,x)) = \varphi(gh,x)$ $\left(\text{or } \varphi(g,\varphi(h,x)) = \varphi(hg,x)\right)$ for any $x \in X$ and $g,h \in G$,

then G is said to *act* on X *on the left* (or *on the right*, respectively), and is called a *transformation group on* X. Usually $\varphi(g,x)$ is denoted simply gx (or xg), if the action is on the left (or on the right).

An action $\varphi : G \times X \to X$ is said to be *transitive* if for any $x, y \in X$ there exists an element g in G such that $\varphi(g,x) = y$. Given $x \in X$, let G_x be the subgroup of G defined by

$G_x = \{g \in G; \varphi(g, x) = x\}$.

This subgroup G_x is called the *isotropy subgroup* (or *stabilizer*) at $x \in X$. The action φ is said to be *effective* (or *faithful*), if $\bigcap_{x \in X} G_x = \{e\}$, and said to be *free*, if $\bigcup_{x \in X} G_x = \{e\}$. The subset $\{\varphi(g, p); g \in G\} \subset X$ is called the *orbit* of a point p, and is denoted by Gp, if the action is on the left, and by pG, if it is on the right.

Multiplications of matrices and vectors induce actions of the general linear group $GL_n(\mathbb{R})$ and of subgroups of $GL_n(\mathbb{R})$ on the space of vectors. If the vectors are columns, the action is on the left, and if rows, the action is on the right.

If a group *G* acts on a set *X*, and *Y* is a subset of *X*, then subgroup G_Y of elements $g \in G$ which transforms *Y* onto *Y* is called the *symmetry group* of *Y*.

2.3. Lie Group

The general linear group $GL_n(\mathbb{R})$ is an open subset in the linear space of matrices $M_n(\mathbb{R})$. Therefore it is an n^2-dimensional manifold. The group operations of multiplication and inversion are differentiable maps of class C^∞ with respect to the differentiable structure of $GL_n(\mathbb{R})$.

A *Lie group* is a group which also a finite-dimensional C^∞ manifold, where the group operations, multiplication and inversion, are differentiable maps of class C^∞. Examples of Lie groups are matrix groups, such as projective group, affine group, orthogonal group and Euclidean group. They are represented as subgroups of the general linear group and as sub-manifolds of the linear space of matrices.

3. Euclidean Space

The Cartesian coordinate space $\mathbb{R}^n$ is the space of *n*-tuples of real numbers $(x_1, \ldots, x_n)$, which is considered as a linear space over the real number field $\mathbb{R}$ with the vector space operations

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$
$$a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n).$$

In the following, $\mathbb{R}^n$ is identified with the space of column vectors of size *n*, where the general linear group $GL_n(\mathbb{R})$ acts on the left, so that for a matrix $g \in GL_n(\mathbb{R})$ and a point $p \in \mathbb{R}^n$, gp denotes the point in $\mathbb{R}^n$ obtained by multiplication of a matrix and a column vector.

3.1. Euclidean Vector Space

A positive definite inner product of a finite-dimensional real vector space is sometimes called a *Euclidean inner product*. A *Euclidean vector space* $\left(V,\langle,\rangle\right)$ is, by definition, a finite-dimensional real vector space V equipped with a positive definite inner product $\langle,\rangle$. The *norm* is a real valued function on V defined by $v \mapsto \| v \| = \sqrt{\langle v, v\rangle}$, and the value $\| v \|$ is called the *length* of a vector v. Norm satisfies *Cauchy-Schwarz inequality*

$$\left|\langle v, w\rangle\right| \le \|v\| \cdot \|w\|$$

for any $v, w \in V$, where the equality holds if and only if v and w are linearly dependent. This inequality allows to define the *internal angle* θ with $0 \le \theta \le \pi$ of two nonzero vectors $v, w \in V$ by the equation $\langle v, w\rangle = \|v\| \cdot \|w\| \cos\theta$.

An *isomorphism* between two Euclidean vector spaces $\left(V_1, \langle,\rangle_1\right)$ and $\left(V_2, \langle,\rangle_2\right)$ is a linear isomorphism $\varphi : V_1 \to V_2$ that satisfies

$$\left\langle \varphi(v), \varphi(w)\right\rangle_2 = \langle v, w\rangle_1,$$

for any $v, w \in V_1$. A linear basis of a Euclidean vector space is called an *orthonormal basis*, if it is composed of mutually perpendicular unit vectors. Gram-Schmidt process of orthogonalization affirms the existence of an orthonormal basis on any Euclidean vector space, and thus the existence of an isomorphism between two Euclidean vector spaces of the same dimension. Actually, if $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are orthonormal bases of V_1 and V_2 respectively, then the linear map $\varphi : V_1 \to V_2$ that assigns v_i to $u_i\left(i = 1, \ldots, n\right)$ is an isomorphism between them.

In the following, $\mathbb{R}^n$ is regarded as a Euclidean vector space by virtue of the standard inner product (the *dot product*); for $x = \left(x_1, \ldots, x_n\right), y = \left(y_1, \ldots, y_n\right) \in \mathbb{R}^n$,

$$\left\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\right\rangle = x \cdot y = x_1 y_1 + \ldots + x_n y_n.$$

An $n \times n$ square matrix X of real entries is called any *orthogonal matrix*, if and only if the transposed matrix X^{T} is the inverse matrix X^{-1}. The set of all orthogonal matrices of degree n is denoted $O_n(\mathbb{R})$, which forms a Lie group of dimension $n(n-1)/2$ called an *orthogonal group*. All orthogonal matrices have determinant ± 1, and the set of orthogonal matrices with determinant equal to 1 forms a subgroup $SO_n(\mathbb{R})$, which is called a *special orthogonal group*. $O_n(\mathbb{R})$ is a compact topological space, and has two connected components. $SO_n(\mathbb{R})$ is the connected component that contains the unit matrix.

The set of vectors of unit length is called the *unit sphere*, and denoted by S^n. S^1 is called the *unit circle*; $S^n = \left\{ v = \left(v_0, v_1, \ldots, v_n\right) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} v_i^2 = 1 \right\}$.

Let $v = \left(v_1, v_2, v_3\right)$, $w = \left(w_1, w_2, w_3\right)$ be vectors in $\mathbb{R}^3$. A vector $v \times w$ defined by

$$v \times w = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, \begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right).$$

is called the *vector product* (also called the *cross product*) of *v* and *w*. This binary operation is useful for three-dimensional Euclidean geometry, and has the following properties. The vector product $v \times w$ is perpendicular to *v* and *w*, and has length equal to the area of the parallelogram spanned by *v* and *w*. If *v* and *w* are linearly independent, then the determinant $\det\left(v, w, v \times w\right)$ is positive. (Here each point in $\mathbb{R}^3$ is regarded as a column vector, so that $\left(v, w, v \times w\right)$ is a 3 × 3 matrix). These three conditions completely determine the vector product.

The vector product also satisfies the following:

- the assignment $\mathbb{R}^3 \times \mathbb{R}^3 \ni \left(v, w\right) \mapsto v \times w \in \mathbb{R}^3$ is bilinear and anti-symmetric,
- $\det\left(u, v, w\right) = \left\langle u, v \times w \right\rangle$,
- $\left\langle u \times v, w \times x \right\rangle = \left\langle u, w \right\rangle \left\langle v, x \right\rangle - \left\langle u, x \right\rangle \left\langle v, w \right\rangle$,
- $\left(u \times v\right) \times w = \left\langle u, w \right\rangle v - \left\langle v, w \right\rangle u$
- $u \times \left(v \times w\right) + v \times \left(w \times u\right) + w \times \left(u \times v\right) = 0$
- $\left\| v \times w \right\|^2 + \left| \left\langle v, w \right\rangle \right|^2 = \left\| v \right\|^2 \left\| w \right\|^2$,

for any vector $u, v, w, x \in \mathbb{R}^3$.

For a real number α, let R_α denote the matrix defined by

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Then the transformation $\mathbb{R}^2 \ni v \mapsto R_\alpha v \in \mathbb{R}^2$ represents a rotation about the origin through the angle α, and set of all R_α with $0 \le \alpha < 2\pi$ is equal to $SO_2\left(\mathbb{R}\right)$. $SO_2\left(\mathbb{R}\right)$ is an Abelian group, and topologically homeomorphic to the circle S^1.

In $\mathbb{R}^3$, for any pair $\left(v_1, v_2\right)$ of mutually perpendicular vectors with unit length, the matrix $(v_1, v_2, v_1 \times v_2)$ is an orthogonal matrix with determinant equal to 1. The map $(v_1, v_2, v_1 \times v_2) \mapsto v_1$ gives a fiber space structure of $SO_3(\mathbb{R})$ over the unit sphere S^2 with fiber homeomorphic to the circle S^1. (For fiber space, see *Topology*) $SO_3(\mathbb{R})$ is a noncommutative group and topologically homeomorphic to the real projective space of dimension three. In general $O_n(\mathbb{R})$ is a Lie group of dimension equal to $n(n-1)/2$. As a topological space, $SO_n(\mathbb{R})$ is obtained by successive constructions of fiber spaces starting with S^{n-1}, and adding S^i each time as fiber for $i = n-2, \dots, 1$ in turn.

If one represents the rotation in the *x*-axis through the angle α by the matrix $X_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & R_\alpha \end{pmatrix}$, and the rotations in the *y*-axis and *z*-axis through angles β and γ by the matrices Y_β, Z_γ similarly defined, then the product $Z_\gamma Y_\beta X_\alpha$ is an element of $SO_3(\mathbb{R})$, and the triplet (α, β, γ) is called the *Euler angle* of this matrix. All matrices in $SO_3(\mathbb{R})$ are obtained in this way with some angles α, β and γ.

3.2. Euclidean Space

Define a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(x, y) = \|x - y\|$$

for any $x, y \in \mathbb{R}^n$. Then d satisfies the three conditions of distance function (see Section 2), and thus $\mathbb{R}^n$ is regarded as a metric space. The metric space $(\mathbb{R}^n, d)$ so obtained is denoted by $\mathbb{E}^n$ and is called the *n*-dimensional *Euclidean space*. The function *d* is called the *Euclidean distance*. A metric space isometric to $(\mathbb{R}^n, d)$ is also called a Euclidean space, and denoted by $\mathbb{E}^n$. Two-dimensional Euclidean spaces are usually called *Euclidean planes*.

An image of isometry from $\mathbb{E}^k$ to $\mathbb{E}^n$ is affine subspace in $\mathbb{E}^n$. In Euclidean geometry, an affine subspace is called a *Euclidean subspace*. One- or two-dimensional Euclidean subspace is called a *line* or a *plane*, respectively. For any two points in $\mathbb{E}^n$, there exists a unique line that contains those two points. For any three points that are not on a line, there exists a unique plane that contains those three points.

An (*n* -1)-dimensional Euclidean subspace in $\mathbb{E}^n$ is called *hyperplane*. A subset defined by a linear inequality $a \cdot x \le c$ (or $a \cdot x < c$) is called a *closed* (or *open*, respectively) *half space*, and the hyperplane $a \cdot x = c$ is called the *boundary*, where *a* is a non-zero vector in $\mathbb{E}^n$ and *c* is a constant number.

A subset X of $\mathbb{E}^n$ is called *convex*, if and only if for any two points p and $q \in X$ the line segment between p and q is contained in X. A half space is convex. The intersection of two convex sets is convex. For a subset X the intersection of all half spaces that contain X is called the *convex hull* of X. For a finite set X the convex hull of X is called a *polytope*, provided X is not contained in any hyperplane. A polytope in $\mathbb{E}^2$ or in $\mathbb{E}^3$ is called a *polygon* or a *polyhedron*, respectively.

For three points $A, B, C \in \mathbb{E}^n$, the angle ABC means the internal angle between the vectors $A - B$ and $C - B$. The notation AB means both the line segment terminating at A and B and its length.

Since the distance function satisfies the triangle inequality, the *triangle inequality*
$AB + BC > AC$
holds for any triangle ABC. Conversely, if three positive numbers a, b and c satisfy the inequalities $a + b > c$, $b + c > a$ and $c + a > b$, then there exists a triangle with the sides of lengths equal to a, b and c.

The following are fundamental theorems of Euclidean geometry:

Theorem 1. The internal angles $\theta_1, \theta_2, \theta_3$ of a triangle satisfy the equality $\theta_1 + \theta_2 + \theta_3 = \pi$.

Theorem 2 (Pythagorean theorem).If a triangle ABC has a right angle at A, the equality $AB^2 + AC^2 = BC^2$ holds.

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Biographical Sketch

Name: **Tetsuya OZAWA** Birth: 1954 in Nagoya, Japan Education:BS in Mathematics, Nagoya University, Japan (March, 1977).MS in Mathematics, Nagoya University, Japan (March, 1979).Doctorat de 3eme cycle in Mathematics, Paris VII, France (June, 1984).Positions held:

Assistant Associe, U.E.R. de Mathematiques, Paris VII, France (February, 1984--June, 1984)Assistant, Department of Mathematics, Nagoya University, Japan (July, 1984--March, 1990)Instructor, Department of Mathematics, Nagoya University, Japan (April, 1990--March, 1995)Assistant Professor, Department of Mathematics, Meijo University, Japan (April, 1995--March, 1998)Professor, Department of Mathematics, Meijo University, Japan (April, 1998 to date)