ALTERNATIVE PROBABILISTIC SYSTEMS

Kurt Weichselberger
Munich, Germany

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Contents

Summary

The subject of this article is probabilistic systems generalizing the usual – "classical" – probability theory and requiring less information to formulate a probability statement. An important example of such a system is provided by interval-valued probability allowing for $P(A) = [0.2;\ 0.3]$ and similar propositions. While from a theoretical standpoint all empirical knowledge must be of restricted accuracy, the practical relevance of such systems can be recognized in many fields:

(1) in classical statistics for instance, as soon as confidence regions are to be included in further models;
(2) in artificial intelligence, as soon as it is realized that the experts' knowledge is not sufficient to justify classical probabilities;
(3) in insurance, if the inhomogeneity of the ensembles used is considered;
(4) in science and technology, as soon as the inevitable measurement errors are taken into consideration;
(5) in all behavioral sciences, if modeling is to be realistic.

Up to now a considerable number of approaches – partially comprehensive – appeared to meet these requirements. However, in large parts of the scientific community the problems described are neglected or met by short-cut methods. In the near future growing attention to this subject can be expected.

1. Introduction

During the last decades of the 20th century a considerable number of proposals were made to support the customary ("classical") theory of probability by other methods of describing and handling situations of uncertainty. For all of these proposals the justifications are similar, arguing that in many situations of practical relevance the requirements for an employment of classical probability calculus are not given.

Three classes of proposed methods to deal with uncertainty can be distinguished:

1. methods avoiding the concept of probability altogether;
2. methods which originally rely on the concept of probability, but eventually contradict the classical theory;
3. methods based on generalizations of the classical theory; in suitable circumstances they become classical ones.

While it is obvious that systems producing methods of class 3 have to be included under the heading "Alternative Probabilistic Systems" and not the concepts related to methods of class 1, those systems on which methods of class 2 rely must be seen as border cases. Therefore their description will be distinguished form that of systems containing classical theory as a special case.

Consequently in this article neither the systems MYCIN and E-MYCIN are described nor those based on fuzzy logic and employing the concept of membership functions. On the other hand the methods of classical statistics to gain evidence about membership functions on the basis of random sampling may be regarded as a kind of alternative probabilistic system. This methodology was described by Viertl (1996).

Beyond the scope of the present contribution remain the deviations between the concept of classical probability on one side and the concepts of probability as used by quantum physicists on the other. Only recently one of them admitted: "Of course, quantum probability calculus gives useful and convenient description of quantum phenomena. However, quantum probability has no direct connection with probability. This is just rather speculative use of the word 'probability' in some formal mathematical constructions." (Khrennikov, 1999, p. 7)

2. Early developments

Already in the 19th century doubts were expressed whether classical probability can be seen as the sole means of describing situations of uncertainty (Boole Peirce). About 1920 J.M. Keynes as well as F.H. Knight gave strong arguments against the traditional attitude: Keynes emphasized the possibility of hypotheses h and k, where neither h is more probable than k nor k is more probable than h nor both are equal in probability. Knight with respect to economic situations distinguished the concepts of risk and uncertainty, employing the random draw of a ball out of an urn as an example for a situation of risk if the composition of the urn is known exactly, and as an example for the situation of uncertainty if this is not true.

During the Second World War two concrete proposals of using imprecise probabilities were made. In 1940 and 1941 B.O. Koopman presented the concept of interval-valued probability by defining upper and lower probabilities for conclusions via comparisons of their probability with those showing known classical probability. Often his articles are regarded as the origin of calculus of interval-valued probability. Quite independently in 1943 E. Borel in his book “Les Probabilités et la Vie” discussed the conditions of offering a wager and accepting a wager under realistic situations. If the probability of the crucial event A cannot be determined exactly, he argued, it is rational behavior to use the lowest value of this probability when deciding whether to accept the wager on A, but to rely on the highest one when fixing the odds at which to offer a wager on A. The influence of Borel’s conception can be seen in many contributions to come.

During the early 1960s interest in lower and upper probability grew remarkably. In 1960 C.A.B. Smith gave a lecture to the Royal Society, which was printed in 1961. Following the line of Borel he defined “lower pignic odds” as the upper bounds on the odds leading to acceptance of the bet and “upper pignic odds” as the smallest ones allowing the offer of this bet. Pignic odds determine lower and upper probabilities and therefore they define the set of “medial probabilities” between: interval-valued probability. In 1962 I.J. Good took up Koopman’s approach applying it to personalist probability of an event: Due to necessary limitations of our knowledge only upper and lower constraints to the “perfect” value of the probability can be given. His system of axioms is similar to that of Koopman.

Since 1961 the motivation to employ interval-valued probability was grounded on fundamental reasoning by Henry E. Kyburg Jr.: Rational belief must be determined objectively, and empirical belief therefore must be based on statistical knowledge which never can be precise. Even strongest possible statements must contain lower and upper limits of probability. Since 1961 Kyburg stressed this argument several times.

Of special importance for the treatment of uncertainty in microeconomics was the report given by D. Ellsberg in 1961. Reactions to L.J. Savage’s “Foundations of Statistics” of 1954 new discussions among economists came up about the use of (classical) probability in describing any situation of uncertainty. Ellsberg undertook testing the practicability of Savage’s axioms under special conditions. In one of his experiments he asked economists to choose between two different settings: A prize was to be given, if a ball randomly selected from an urn had a certain color – but there was partially restricted knowledge about the composition of the balls in the urns. He reported, that situations exist, under which a vast majority of economists through their decisions violate the rules of classical probability, especially the law of additivity and Savage’s “sure thing-principle”. This result is widely known under the name “Ellsberg-paradox”. He proposed the expression “ambiguity” to characterize aspects responsible for this violation – like that caused by the lack of information on the exact composition of the urn in his experiment. In the sequel the concept of ambiguity has provoked a series of theoretical as well as empirical investigations.

Most of Ellsberg’s results may be explained by assuming that an agent rationally decides, as if among all classical probability assessments not explicitly excluded by his information that one were true which is worst for his interest. This strategy can be identified with the application of the minimax-principle to the set of possible classical probability assessments. It is in agreement with Borel’s recommendation for behavior in dealing with a wager and Smith’s concept of pignic odds. This concept was seen as a promising decision-theoretical approach to imprecise probability. It was named Γ-minimax-strategy and described by many authors.

On the other hand, Ellsberg himself had warned against assuming general validity of this principle. His simple and convincing argument rests on the description of an example where there is choice between two settings, each of them promising the same prize provided that a random draw out of an urn produces a red ball. One of the settings employs urn 1 with a composition totally known: The proportion of red balls is $p > 0$. The other uses urn 2, where the composition is unknown: The proportion of red balls can have any value between 0 and 1. For every $p > 0$, the Γ-minimax strategy requires to choose the setting with urn 1, since in the worst case for the decision-maker there are no red balls at all in urn 2. But Ellsberg stresses that every one in fact will switch to the setting with urn 2, provided that p becomes sufficiently small. Under the given circumstances choosing urn 1 would make it practically sure: no red ball, no prize. Choosing the other setting leaves the hope that the composition of urn 2 is more favorable for the decision-maker. This example demonstrates that none obeys Γ-minimax-strategy unconditionally. For many years Ellsberg's warning appeared to have been generally neglected, although there exist large classes of strategies explaining his results at least as well as Γ-minimax does

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Biographical Sketch

Kurt Weichselberger, born 1929 in Vienna. Dr. phil. 1953 (Univ. Vienna, Mathematics and Physics). Statistician in Social Research. Venia legendi 1962 (Univ. Cologne), Privatdozent. Guest-professor 1963 (Univ. Göttingen). Full professor: 1963-1969 (Technical University Berlin, Rektor 1967-1968), 1969- (Univ. Munich, emerit. 1997). Research: Methodology, Interval-probability.