PROBABILISTIC MODELS AND METHODS

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Summary

In this chapter, we discuss mathematical methods to analyze problems involving uncertainty. Such problems arise naturally in many fields including economics, finance, biology, medical decision making and operations research. Probability and stochastic processes are useful mathematical tools for that purpose. This article explains useful notions of probability and stochastic processes with typical applications.

1. Introduction

The ideas of probability are around us. Lotteries and gambling are direct examples in which the principles of probability are used to examine the processes and outcomes. For example, consider a simple game in which you can earn 1 dollar if you have an even number by rolling a die. The problem of interest may be how much you should pay to enter the game. In order to answer this question, we need to construct a mathematical model to describe the problem. A probabilistic model would be useful for that purpose, because the game involves uncertainty and probability is a mathematical tool to describe such a situation. In fact, probability as a formal theory has begun in the 17th century with a correspondence between the two famous French mathematicians, B. Pascal and P. Fermat, about appropriate odds of a card game.

The aim of this article is to provide mathematical methods, namely probability and stochastic processes that are used to analyze problems involving uncertainty. Such problems arise naturally in many fields including economics, finance, biology, medical decision making and operations research. Each section explains useful notions of probability and stochastic processes with typical applications. It is not intended to give mathematically rigorous exposition of those notions, but emphasis is put on showing the usefulness of such mathematical methods when analyzing random phenomena.

2. A Simple Probabilistic Model

In this section, we consider a simple probabilistic model from economics to introduce some important ideas of probability. To do so, suppose we are given the above problem of rolling a die and try to answer the question how much you should pay to enter the game.

The first step in making the question precise is to define a *random variable* that represents possible outcomes together with a certain probability distribution. Namely, let X be the money you earn, and let p_i be the probability that the rolled die appears to have number i. The random variable X in this case is formally defined as

$$X = \begin{cases} 1, & i = 2, 4, 6, \\ 0, & i = 1, 3, 5. \end{cases}$$

What is crucial here is that you can calculate the probability of each outcome in X based on the given probability distribution $\{p_1, p_2, \ldots, p_6\}$. In this case, we obtain

$$P\{X = 1\} = p_2 + p_4 + p_6 \text{ and } P\{X = 0\} = p_1 + p_3 + p_5 .$$

If you believe that the die is a fair one, then you may assume that the probability of having number i is equally likely, i.e. $p_i = 1/6$ for all i. However, you can assume another distribution, called *subjective probability*, as far as it satisfies the mathematical axioms: (i) $p_i \geq 0$ for all i, and (ii) $\sum_{i=1}^{6} p_i = 1$. In any case, a random variable is defined by enumerating possible outcomes and by assigning each outcome an appropriate probability that satisfies the technical conditions.

Given the random variable X and a real-valued function $u(x)$, the *expectation* is defined by

$$E\left[u(X)\right] = \int_{-\infty}^{\infty} u(x)\, dF(x), \tag{1}$$

for which the integral exists, i.e. $\int_{-\infty}^{\infty}\left|u(x)\right|dF(x)<\infty$. Here, $F(x)= P\{X \le x\}$ is called the *distribution function* of X and the integral is, in the most general setting, the Lebesgue-Stieltjes integral. If $F(x)$ is differentiable, the expectation is given by

$$E\left[u(X)\right]=\int_{-\infty}^{\infty}u(x)\,f(x)dx,$$

where $f(x)= dF(x)/dx$ is called the *probability density function* of X. In the above discrete case, however, the integral is simply reduced to a weighted sum,

$$E\left[u(X)\right]=\sum_{i=1}^{6}u(i)\,p_i\ .$$

Especially, when $u(x) = x$, the expectation $E[X]$ is often called the *mean* of X. If $u(x) = (x - E[X])^2$, we have

$$V\left[X\right]\equiv E\left[\left(X-E\left[X\right]\right)^2\right]=E\left[X^2\right]-\left(E\left[X\right]\right)^2\ , \qquad (2)$$

called the *variance* of X. The square root of variance, $\sigma_X=\sqrt{V\left[X\right]}$, is called the *standard deviation*. The standard deviation (or the variance) is used as a measure of spread or dispersion of the distribution around the mean. The bigger the standard deviation, the wider the spread, and there is more chance of having a value far below (and also above) the mean.

The most important property of expectation is *linearity*. That is, for random variables X and Y and constants a and b, we have

$$E\left[aX+bY\right]=aE\left[X\right]+bE\left[Y\right]\ , \qquad (3)$$

provided that the expectations exist. Notice that $aX + bY$ is also a random variable. Hence, it is possible to calculate the distribution function of $aX + bY$, $G(x)$ say, from the given distribution functions of X and Y, $F_X(x)$ and $F_Y(x)$ say, respectively. The linearity expression (3) reveals that the mean of $aX + bY$, calculated in terms of $G(x)$, is equal to the weighted sum of the means, $E[X]$ and $E[Y]$, calculated in terms of $F_X(x)$ and $F_Y(x)$, respectively.

As an example, suppose there is another participant to the above game. Suppose that the money he/she can earn is the random variable,

$$Y=\begin{cases}1, & i=1,2,3,\\ 0, & i=4,5,6.\end{cases}$$

Then, the expected total money that either of the two can earn is given by

$$X+Y=\begin{cases}2, & i=2,\\ 1, & i=1,3,4,6,\\ 0, & i=5.\end{cases}$$

It follows that

$$E\left[X+Y\right]=2p_2+\left(p_1+p_3+p_4+p_6\right).$$

On the other hand, we obtain $E[X]= p_2 + p_4 + p_6$ and $E[Y]= p_1 + p_2 + p_3$, and hence the linearity (3) is confirmed. Notice that the second equality in (2) can also be proved by the linearity of expectation. That is, denoting $\mu = E[X]$, we have

$$E\left[\left(X-\mu\right)^2\right]=E\left[X^2\right]-2\mu E\left[X\right]+\mu^2=E\left[X^2\right]-\mu^2 \quad .$$

We are now in a position to answer the question how much you would pay to enter the game. Notice that the expected value of the amount that you can earn is $E[X]$. So, a plausible answer for the fair bet may be $E[X]$, since you can expect this amount of money before you enter the game. However, you earn nothing and lose the bet when the die comes up in an odd number. If you do not enter the game, you possess that money for sure, which can be used for other purposes. There exists an investor, called *risk-averter*, who prefers the money for sure to uncertain gambling if the expected outcome of the gamble is the same as the sure money. If you are risk-averse, then the theory of *expected utility principle* developed by von Neumann and Morgenstern (1944) is used to evaluate the value of the game. Let $u(x)$ denote your utility function, i.e. you assign the value $u(x)$ to outcome x. The theory tells that you evaluate the game to be $E[u(X)]$, and you enter the game if and only if the bet is less than or equal to the expectation $E[u(X)]$.

The expected utility principle is closely related to the notion of *stochastic orders* which have been developed in applied probability and statistics. Let $\mathcal{F}$ be a class of functions defined on the real line $\mathbb{R}=\left(-\infty,\infty\right)$. For random variables X and Y, a stochastic ordering relation $X\succeq Y$ is said to be generated from $\mathcal{F}$ if

$$E\left[u\left(X\right)\right]\geq E\left[u\left(Y\right)\right] \quad \text{for all } u\left(x\right)\in\mathcal{F} \tag{4}$$

for which the expectations exist. In order to describe (4) in an economic perspective, consider an investor with a von Neumann-Morgenstern utility function $u(x)$. Then, $X\succeq Y$ if and only if all investors whose utility functions belong to the class $\mathcal{F}$ prefer X to Y. In economics, investors are assumed to prefer more to less and risk-averse. Hence, a candidate for the function class is

$$\mathcal{F}=\left\{u\left(x\right):u\left(x\right)\text{ is increasing and concave in } x\right\} \tag{5}$$

The stochastic order generated from this class of functions is called the *second order stochastic dominance* in economics, or the *concave order* in applied probability.

3. Risk Management

The standard deviation is used as a measure of risk in financial economics. For example, in the portfolio selection technique developed by Markowitz (1959), an optimal portfolio is determined by minimizing the standard deviation for a given expected return. To be more specific, let X_i denote the random variable representing the future return of security *i*. A *portfolio* is a random variable of the form,

$$P = \sum_{i=1}^{n} a_i X_i, \quad (a_1,\ a_2,\ ....,\ a_n) \in \mathcal{A},$$

where $\mathcal{A}$ denotes some constraint. If X_i are normally distributed (see (7) below for the definition), then the portfolio *P* is also normally distributed, since a linear combination of normally distributed random variables is normally distributed. Also, since the probabilistic property of normally distributed random variable *P* is determined by the mean $\mu_P \equiv E[P]$ and the standard deviation $\sigma_P \equiv \sqrt{V[P]}$, the expected utility $E[u(P)]$ is also a function of μ_P and σ_P. It is known that, for $u(x) \in \mathcal{F}$ in (5), $E[u(P)]$ is increasing in μ_P, while it is decreasing with respect to σ_P. The problem is thus formulated so as to minimize σ_P subject to $\mu_P = \mu$ for a given target μ. The Markowitz problem is a quadratic programming, which can be solved by a mathematical programming technique.

Of more interest for financial institutions in risk management is the potential for significant loss in their portfolios. Namely, given a probability level α, what would be the maximum loss in a portfolio? The maximum loss is called the *value at risk* (VaR) with confidence level 100α%, and VaR is now a very popular tool for risk management. The popularity of VaR is based on aggregation of several components of market risk into a single number.

Let Q_0 be the current market value of a portfolio, and let Q_1 be its future market value (random variable). Assuming that no rebalance will be made, VaR with confidence level 100α% of the portfolio is defined by z_α that satisfies

$$P\{Q_1 - Q_0 \geq -z_\alpha\} = \alpha,\ z_\alpha > 0. \tag{6}$$

Let *R* denote the return of the portfolio, i.e. $R = (Q_1 - Q_0)/Q_0$. In the actual financial market, the probability distribution of the return can be estimated by some statistical means. It follows that the VaR is obtained from $P\{R \geq -z_\alpha / Q_0\} - \alpha$. That is, the value z_α/Q_0 is the 100α-percentile in statistics.

If R is normally distributed, as often assumed in the theory of finance, then the calculation of VaR becomes much simpler. A random variable is said to be *normally distributed* with mean μ and variance σ^2 if its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad x \in \mathbb{R} \tag{7}$$

If this is the case, we denote $X \sim N(\mu, \sigma^2)$. In particular, $N(0, 1)$ is called the *standard* normal distribution. The normal density is symmetric about the mean and bell-like shaped. Also, when $X \sim N(\mu, \sigma^2)$, the transformation

$$Y = \frac{X-\mu}{\sigma} \tag{8}$$

is called the *standardization* of X, and the random variable Y follows $N(0, 1)$. Conversely, it can be shown that if $Y \sim N(0,1)$ then $\mu + \sigma Y \sim N(\mu, \sigma^2)$. These facts are important properties of normal distributions.

Now, suppose that $R \sim N(\mu, \sigma^2)$. In the most of standard textbook in statistics, the table of the survival probabilities of $N(0, 1)$,

$$L(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \qquad x > 0,$$

is provided. Then, using standardization (8) and symmetry of the density function, we can obtain the value z_α in (6) with ease. Let $r_\alpha = -z_\alpha/Q_0$. It follows that

$$1-\alpha = P\left\{\frac{R-\mu}{\sigma} \le \frac{r_\alpha - \mu}{\sigma}\right\} = L\left(-\frac{r_\alpha - \mu}{\sigma}\right).$$

Therefore, letting x_α be such that $L(x_\alpha) = 1 - \alpha$, we obtain $z_\alpha = Q_0(x_\alpha \sigma - \mu)$, as the VaR with confidence level 100α%. The value x_α is called the 100 $(1 - \alpha)$-percentile of the standard normal distribution.

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Biographical Sketch

Masaaki Kijima was born on 31st March 1957 in Japan. He received the D.Sc. degree from the Department of Information Sciences, Tokyo Institute of Technology, Tokyo, in 1985 and the Ph.D. degree from the Graduate School of Management, Rochester University, USA in 1986. He is presently Professor, Graduate School of Economics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-8501, Japan. Dr. Kijima has over 70 publications including the books: (2002), *Stochastic Processes with Applications to Finance*, Chapman & Hall, London, and (1997), *Markov Processes for Stochastic Modeling*, Chapman & Hall, London.