MATHEMATICAL MODELS OF PLASMA PHYSICS

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Summary

Plasma is a partially or fully ionized gas which satisfies the condition of quasi-neutrality. A major part of the universe exists in the state of plasma. Plasma is widely used in industrial and laboratory conditions. In the second part of the 20th century plasma physics was formed as an original branch of physics. The theoretical basis of plasma physics is found in equations of mechanics of continuous media taking into account electromagnetic forces and Maxwell's equations. Different simplifications of these equations give a series of mathematical models. They describe various, complicated processes in plasmas whose spatial and time scales differ by many orders.

1. Introduction

Plasma (from Greek πλασμα, literally, - generated, moulded) is a partially or fully ionized gas, which satisfies the condition of quasi-neutrality. The term "plasma" was introduced in 1923 by American physicists Langmuir and Tonks. A major part of the universe exists in the state of plasma: galactic nebula, stars, interstar medium, magnetosphere and ionosphere enclosing the Earth. Plasma is widely used in industrial and laboratory conditions: various gas discharges, magnetohydrodynamic generators, plasma accelerators, high-temperature plasma in devices designed for controlled thermonuclear fusion.

The properties of plasma essentially differ from those of the usual gases. It is due to two of its singularities. At first, plasma is strongly affected by electric and magnetic fields. They can be divided on exterior and interior. The latter are formed by charges and currents in the plasma. Such peculiar self-action produces a lot of specific properties, related to plasma oscillations and instabilities. As a typical example it is possible to mention longitudinal Langmuir oscillations with frequency $\omega_0=(4\pi ne^2/m_e)^{1/2}$. Secondly, the interaction between charged particles of plasma is determined by Coulomb force with a slowly decaying potential. Due to this the basic contribution to changes of a distribution function of particles over velocities is given by far collisions, at which the magnitude of transmitted impulse $\Delta\mathbf{p}$ is small. As a result of this the Coulomb collision operator, specific for plasma, differs from the classical Boltzmann collision integral for gases.

Plasma is characterized by a large number of parameters. The ratio of the number of ionized atoms to their total number is called the degree of ionization. Plasma can be in weak, strong and fully ionized states. The degree of ionization depends on the temperature and exterior action, for example, on a radiation flow. The simplest is the case of the fully ionized plasma. Such a state can be obtained only for the lightest elements from hydrogen up to carbon. The equilibrium composition of the weakly ionized plasma can be calculated well enough using the Saha formula. The most complicated for description is the case of plasma of heavy elements. The multicharged ions are present in it having a different degree of ionization and maintaining a part of their electrons.

The temperature of plasma varies over a wide range depending on its origin. Plasma with $T \leq 10^5$ K is considered as the low-temperature, with $T \approx 10^6 - 10^8$ K – as the high-temperature. For ignition of controlled thermonuclear fusion with positive balance of energy it is necessary to heat up deuterium-tritium mixture to a temperature exceeding 10^8 K. In many cases plasma can be nonisothermal, then it is necessary to distinguish temperature of electrons T_e, ions T_i, and non–ionized atoms T_a.

The essential distinction of electron and ion masses results in various characteristic times of relaxational processes and establishment of Maxwellian distribution functions of particles of different sorts. In the elementary case of homogeneous, fully ionized plasma consisting of electrons and one-charge ions ($n_e = n_i = n$) it is possible to choose four characteristic times: τ_e is the time of a maxwellization distribution function of electrons as a result of their collisions, τ_i is a similar characteristic time for ions, τ_{ei} is the relaxation time of relative motion of electrons and ions, τ_T is the characteristic time of energy exchange between electrons and ions in nonisothermal plasma. Accepting the fastest time τ_e as the basic, one can determine the following hierarchy between the characteristic times:

$$\tau_e = \frac{(kT_e)^{3/2}\sqrt{m_e}}{\sqrt{2\pi}\, e^4 nL}, \quad \tau_{ei} \approx \tau_e, \quad \tau_i \approx \sqrt{\frac{m_i}{m_e}}\,\tau_e, \quad \tau_T \approx \frac{m_i}{m_e}\tau_e. \qquad (1)$$

Here e is the elementary charge, m_e and m_i are masses of electrons and ions, n is a plasma density, k is the Boltzmann constant, L is the Coulomb logarithm. Due to major difference in masses the "slowest" process is the process of energy exchange between electrons and ions.

The plasma density varies in much wider limits than temperature. For space mediums the range of plasma densities is of 30 orders: from 10^{-6} cm^{-3} for interstellar space up to $10^{23} - 10^{24}$ cm^{-3} and above in stars. The range of plasma densities produced for different purposes by human is also wide enough. For example, in controlled thermonuclear fusion research the density of plasma varies from $10^{13} - 10^{14}$ cm^{-3} in tokamaks up to $10^{23} - 10^{24}$ cm^{-3} in special targets for laser thermonuclear fusion.

It is necessary in multicomponent plasma to introduce density n_α for every ion component of plasma separately. The ions differ not only by chemical elements but also by the degree of ionization, so the requirement of plasma neutrality becomes:

$$n_e = \sum_\alpha z_\alpha n_\alpha, \qquad (2)$$

where z_α is the multiplicity of the ion charges. If the equilibrium distribution of ions over degrees of ionization in multicomponent plasma is not established, there can be very sharp changes in densities of separate components.

The natural and laboratory plasma in many cases is magnetized. It has a number of specific peculiarities. The Lorentz force makes charged particles to move over complicated trajectories: they freely move along the field lines with velocity $v_{\perp}$, rotate in a plane perpendicular to the field line with Larmor circle of radius r_B with frequency ω_B:

$$\omega_{\mathrm{B}} = \frac{e_{\alpha}B}{m_{\alpha}c}, \quad r_{\mathrm{B}} = \frac{v_{\perp}}{\omega_{\mathrm{B}}}, \tag{3}$$

where $v_{\perp}$ is the component of velocity perpendicular to the magnetic field. At last, the centre of a Larmor circle drifts perpendicular to the magnetic field. The velocity of the drift is defined by the gravitational field, electric field and non-uniformity of the magnetic field. The energy ε and the magnetic moment $\mu = m_{\alpha} v_{\perp}^2 /(2B)$ are conserved during that composite motion.

The essential influence on the character of motion of charged particles in a magnetic field refers not only the local value of its strength, defining the Larmor frequency and radius, but also the general topological structure of its field lines. For example, the field of the Earth looks like a magnetic dipole, its strength increases near the magnetic poles. The auroras in near Earth plasma are interlinked to it. Magnetic confinement and thermo-insulation of high-temperature plasma are provided by the special structure of the field in tokamak.

Plasma is complex and manifold in its appearance. Its behaviour is determined by the processes of diverse nature which have spatial and time scales distinguishing by many orders. Mathematical models are used for plasma description, which include equations of the mechanics of continuous media taking into account electromagnetic forces and Maxwell's equations. The kinetic, magnetohydrodynamic and transport models of plasma are distinguished depending on the chosen approximation.

2. Kinetic models

2.1. Liouville equation

The kinetic models give the most detailed description of gas and plasmas. The following probability representations are their basis. A system, consisting of N particles, is described with the help of the distribution function $F(t, \mathbf{x}_1,...,\mathbf{x}_N)$, where $\mathbf{x}_i=(\mathbf{r}_i, \mathbf{p}_i)$ are coordinates and impulses of the i-th particle. The distribution function is treated as a probability density in $6N$-dimensional phase space, the integral from which is normalized to unity. If all particles make only mechanical motion, so that the number of particles of each kind does not vary (for example, there is no ionization, recombination, chemical transmutations), then one can write for function F the equation of continuity in the phase space and transform it with the help of the Hamilton equations of motion to the form:

$$\frac{\partial F}{\partial t}+\sum_{i=1}^{N}\left(\frac{\partial H}{\partial \mathbf{p}_i}\frac{\partial F}{\partial \mathbf{r}_i}-\frac{\partial H}{\partial \mathbf{r}_i}\frac{\partial F}{\partial \mathbf{p}_i}\right)=0, \qquad (4)$$

where $H(\mathbf{x}_1,...,\mathbf{x}_N)$ is the Hamiltonian function of the considered system of particles. Equation (4) is called the Liouville equation.

2.2. BBGKY hierarchy of kinetic equations

The Liouville equation plays an important role in the construction and justification of the kinetic models of plasma. However because of the large number of variables it is too complicated for solution of practical problems. Integrating the distribution function F in part of its arguments, it is possible to introduce one-particle distribution functions, two-particle distribution functions and so on, and to deduce relevant equations for them.

One-particle distribution function of particles of kind α with argument $\mathbf{x}_1=(\mathbf{r}_1,\mathbf{p}_1)$ is obtained from function F by the following way:

$$f_\alpha(t,\mathbf{x}_1)=\frac{N_\alpha}{V}\int F(t,\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_N,)d\mathbf{x}_2,...,d\mathbf{x}_N. \qquad (5)$$

Here V is the geometrical volume occupied by the plasma, N_α is the total number of particles of kind α, which is used to normalize function $f_\alpha(t, \mathbf{x}_1)$. Two-particle distribution functions, three-partial distribution functions are introduced similarly.

If one integrates the Liouville equation over all variables, except for $\mathbf{x}_1$, the equation for one-particle function $f_\alpha(t, \mathbf{x}_1)$ is obtained, in which the two-particle distribution functions enter in an integral term. In a similar way it is possible to get the equation for two-particle distribution functions $f_{\alpha\beta}(t, \mathbf{x}_1, \mathbf{x}_2)$ containing an integral term with three-particle distribution functions and so on. The obtained engaging chain of equations is known as Bogolubov, Born, Green, Kirkwood, Yvon hierarchy (BBGKY hierarchy). Its construction is appropriated to the decomposition of the Liouville equation in powers of parameter ν, which is a ratio of the mean energy of particle interaction to the mean kinetic energy of particles. BBGKY hierarchy is more convenient for the further analysis, than the Liouville equation.

The basic kinetic models of plasma are obtained from the BBGKY hierarchy with the help of the following additional simplifying assumptions:

- The number of particles is great: $N \gg 1$.
- The energy of interaction between particles is small in comparison with their kinetic energy: ν<<1.
- The action of external fields on the process of conjugate collisions between particles is negligible.

These simplifying assumptions allow to tear off the BBGKY hierarchy and to get the equations for distribution functions for small number of particles. An especially important role in plasma studies is played by the kinetic equations for one-particle distribution functions. The Vlasov equation with the self-consistent electromagnetic field, the Boltzmann equation and the Landau equation with Coulomb collision operator are related to this kind. Historically they have appeared earlier than the BBGKY hierarchy, however the concept of the hierarchy on the basis of the uniform approach has allowed giving them rigorous theoretical ground, determining the regions of applicability, and aiming at ways for construction of a more complicated kinetic models.

2.3. Vlasov equation with the self-consistent electromagnetic field

In 1938 Vlasov proposed the concept of description for a wide range of plasma processes. The basis of the model is in the kinetic equation without the term for collisions:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B})\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0. \qquad (6)$$

Electric and magnetic fields included in the equation through the Lorentz force are determined from Maxwell equations for vacuum:

$$\begin{aligned} &\operatorname{rot}\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \qquad && \operatorname{div}\mathbf{B} = 0, \\ &\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, && \operatorname{div}\mathbf{E} = 4\pi\rho. \end{aligned} \qquad (7)$$

The set of equations is completed by the formulas for the charge and current densities, which are expressed through the distribution function of particles:

$$\begin{aligned} \rho(t,\mathbf{r}) &= \sum_\alpha e_\alpha \int f_\alpha(t,\mathbf{r},\mathbf{v})d\mathbf{v}, \\ \mathbf{j}(t,\mathbf{r}) &= \sum_\alpha e_\alpha \int f_\alpha(t,\mathbf{r},\mathbf{v})\mathbf{v}d\mathbf{v}. \end{aligned} \qquad (8)$$

Here summation over α means summation over all particle species. The field, defined by Equations (7), and (8), is termed as "self-consistent". It is determined by distribution of particles and in its turn influences their motion due to Equation (6). The concept of a self-consistent field appeared to be very effective. The Vlasov equation can be derived from the BBGKY hierarchy with the supposition, that the multiparticle distribution function is the product of one-particle functions.

The set of Equations (6)-(8) has formed the basis for a huge number of papers on a theory of waves, stability, collective processes in plasma. Here are some of the most widely known results:

- Existence in plasma of longitudinal plasma waves. The effect of Landau damping.
- General theory of oscillations and stability of plasma. Effect of the spatial dispersion.

- Stabilization of bump on tail instability at a nonlinear stage by the formation of a plateau on the distribution function. The quasilinear theory of waves in plasma.
- Effect of "echo in plasma".

2.4. Kinetic equation with the operator of binary collisions

The next step in the development of the general concept of the kinetic theory consists in expression of binary correlation function through one-particle distribution functions with the help of simplifying assumptions and in obtaining the kinetic equation with a collision integral:

$$\frac{\partial f_\alpha}{\partial t}+\mathbf{v}\frac{\partial f_\alpha}{\partial \mathbf{r}}+\frac{\mathbf{F}_\alpha}{m_\alpha}\frac{\partial f_\alpha}{\partial \mathbf{v}}=\sum_\beta L_{\alpha\beta}[f_\alpha]. \tag{9}$$

Here $\mathbf{F}_\alpha$ is the exterior force operating on particles of kind α, $L_{\alpha\beta}$ are partial operators of collisions, which describe changes of the distribution function f_α as the result of collisions of particles α with particles β. In particular, the operator $L_{\alpha\alpha}$ describes collisions between particles α. Summation over β means summation over all particles species.

For neutral particles α and β the interaction potential between which quickly decreases with distance the operator $L_{\alpha\beta}$ is the classical Boltzmann collision integral. Equation (9) proposed by Boltzmann underlies the kinetic theory of gases.

The interaction between charged particles submits to Coulomb law. The Coulomb potential slowly decreases with distance. Due to this the basic contribution in the operator $L_{\alpha\beta}$ is given by distant collisions relevant to large aiming parameters. The magnitude of the transmitted impulse $\Delta\mathbf{p}$ is small for them. The account of these singularities of the process results in the operator of Coulomb collisions obtained by Landau:

$$L_{\alpha\beta}[f_\alpha]=\frac{2\pi e_\alpha^2 e_\beta^2 L}{m_\alpha}\frac{\partial}{\partial v_k}\int\left(\frac{f_\beta{}'}{m_\alpha}\frac{\partial f_\alpha}{\partial v_l}-\frac{f_\alpha}{m_\beta}\frac{\partial f_\beta{}'}{\partial v_l{}'}\right)U_{i\kappa}d\mathbf{v}'. \tag{10}$$

Here

$$U_{kl}=\frac{\partial^2|\mathbf{v}-\mathbf{v}'|}{\partial v_k\partial v_l}=\frac{\delta_{kl}}{u}-\frac{u_k u_l}{u^3},\qquad \mathbf{u}=\mathbf{v}-\mathbf{v}', \tag{11}$$

L is the so-called Coulomb logarithm. In 20 years after Landau's work other derivation of the Coulomb collision operator was given. The final result has appeared to be equivalent to (10), but its representation in the form of Fokker-Planck operator is more convenient for solution of practical problems.

The kinetic equations with the Boltzmann and Landau-Fokker-Planck collision operators can be obtained uniformly from BBGKY hierarchy. Both operators have the following properties:

- Reduce to zero for Maxwellian distribution functions of particles α and β with identical temperature and mean transfer velocity.
- Conserve the number of particles.
- Conserve the total impulse of particles α and β.
- Conserve the total energy of particles α and β.
- Do not increment the *H*-function (the Boltzmann *H*-theorem).
- Besides for a Coulomb operator, which is a differential operator of second order in respect to the distribution function f_α, one more property is valid:
- Operators $L_{\alpha\beta}$ are elliptic, i.e. the quadratic form, which can be formed with the help of the matrix of coefficients at the second order derivatives of function f_α in the operator $L_{\alpha\beta}$, is positively defined.

Property 6 reveals the mathematical nature of the Coulomb operator and justifies the correctness of the statement of the problem about the relaxation of the distribution function in the velocity space due to Coulomb collisions.

The operator of Coulomb collisions was widely used for the solution of many problems of plasma physics. Here are references to some examples:

- Determination of characteristic relaxation times.
- Determination of classical plasma conductivity.
- Plasma confinement in adiabatic traps with magnetic mirrors.
- Discovery of the effect of run away electrons.
- Plasma heating by neutral beam injection. Interaction of plasma with thermonuclear alpha-particles.
- Plasma heating by high-frequency electromagnetic fields.

In the next years more complicated kinetic models were developed and applied to research in plasma physics by efforts of Balescu, Lenard, Klimontovich and others.

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Biographical Sketches

Kalitkin N.N., is a Corresponding Member of Russian Academy of Science, Head of Department of Institute of Mathematical Modeling, Prof. of Department of Physics of Lomonosov Moscow State University. His professional interests include the development of mathematical models of materials in extreme conditions and the numerical methods for their solution. The main scientific results are: the quantum-statistical model of compressed hot material was developed that describes the equation of state and shock waves; the models of conductivity and thermodynamic properties of non-ideal plasma were developed; the quasi-band structure of electron spectrums was discovered – it allowed to unite the above described models into single model. On the basis of these models the calculations of a number of physical processes and constructions were performed (the generators of extra-strong magnetic fields, the electric gas discharge lasers).

Kostomarov D.P., is a Corresponding Member of Russian Academy of Science, Prof. of Department of Computational Mathematics & Cybernetics of Lomonosov Moscow State University. His professional interests include the development and investigation of mathematical models of electrodynamics, quantum mechanics and plasma physics. The main results in plasma physics are connected with the research of kinetic, transport and evolutional models and their realization in the form of computer codes and also with the investigation of inverse problems of microwave and corpuscular plasma diagnostics. The methods of their solution have allowed to write the special computer codes for the automated processing of experimental data directed to the determination of density and ion temperature in tokamak devices.