MATHEMATICAL MODELS FOR PREDICTION OF CLIMATE

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Summary

Numerical simulation of global climate models is the major method for examining the changes of contemporary climate under the impact of anthropogenic influences. General circulation models of the atmosphere and ocean are the basis of global models of climate. The development of these models in the last decades was spurred by vigorous development of numerical mathematics and computing facilities.

There is a specific feature of the modeling of climate and especially its future changes: it is impossible to carry out direct physical experiments with the climatic system to determine its sensitivity to small external impacts. It is very important, in this connection, to examine the sensitivity of the climatic system on the basis of the theory of attractors of dissipative semidynamic systems. At present, serious results were obtained in this direction, however many problems remain unsolved. The solution of these problems would allow one to approach the grand problem: the control of climate.

1. Introduction

The prediction of climate changes induced by anthropogenic processes is one of the most important challenges for science in the 21st century. Anthropogenic impacts on the climatic system include the combustion of fossil fuels leading to the increase in the CO_2 concentration in the atmosphere, changes in the concentration of small gaseous species that control the ozone concentration in the atmosphere, deforestation and desertification resulting in changes in albedo, and many other impacts.

Unlike many other problems of physics, these have a distinguishing feature: they do not allow a direct physical experiment. Moreover, adequate laboratory experiments also seem to be questionable because of specific features of the climate system. In fact, from the standpoint of large –scale atmospheric processes, the atmosphere is a thin layer with the ratio between vertical and horizontal scales $\varepsilon = H / L \sim 10^{-3} \div 10^{-4}$; at the same time, vertical distribution of parameters in this layer is very important. Therefore, the method of numerical simulation of global climate models is the main research tool in this case.

We need several definitions to describe these models.

The climatic system is understood as including the atmosphere, ocean, cryosphere, land, and biota.

The state of the climatic system is characterized by a set of distributed parameters: temperature, pressure, humidity, velocity of wind in the atmosphere and of currents in the ocean, concentration of gaseous components, etc.

The climate is an ensemble of states passed by climatic system for a sufficiently long period. Generally speaking, the choice of such a period is a problem: in what follows, we always specify the time interval when we make strict statements.

From a physical standpoint, to examine the climate of the real climatic system, we have only a portion of the trajectory several decades long, during which adequate field measurements were carried out. Naturally, we can restore some characteristics of the climatic system for longer time intervals; however, this concerns only individual characteristics rather than a sufficiently complete set characterizing the state of climatic system.

The construction of modern climate models is based on the following principles.

- The model is constructed so as to sequentially account for all the processes participating in the formation of climate, even if the contribution of some processes to the total energy of the overall process is relatively small. Such an approach is conditioned, in particular, by the fact that the theory of climate's sensitivity to small external impacts is not yet completely developed.
- It is assumed that the Navier–Stokes equations for a compressible fluid describe the dynamics of the atmosphere and ocean (in describing the ocean's dynamics, as a rule, incompressibility is assumed).

- In modern models, by virtue of the computational facilities, the Reynolds equations are employed rather than the Navier–Stokes equations. The Reynolds equations are the Navier–stokes equations averaged over certain temporal and spatial scales observing certain commutation rules for the averaging operators.
- It is assumed that a closure procedure is principally possible at a certain level of accuracy: the expression of subscale processes (the scales smaller than the average scale) through the characteristics of large–scale processes.
- It is assumed that the equations of classic equilibrium thermodynamics are locally valid.
- As a rule, modern climate models use the hydrostatic approximation to describe large-scale atmospheric and oceanic motions: the vertical pressure gradient is compensated by the gravity force. The use of such an approximation involves a number of further simplifications: it is required that the energy conservation law must be satisfied in the absence of external sources of energy and dissipation. In particular, the earth's radius is taken to be constant and the components of Coriolis force with vertical velocity component are disregarded. The hydrostatic approximation reduces the system of three–dimensional Navier-Stokes equations to the system of "2.5 –dimension", which is significant in formulating the theorems concerning unique solvability of these equations on an arbitrarily finite time interval.

The climate model formulated on the basis of these principles (more precisely, its finite–dimensional version) makes it possible to carry out numerical experiments reproducing the contemporary climate and investigate the sensitivity of the "model climate" to small changes in the parameters characterizing the external impacts. However, a question arises: what should the climate model reproduce and with what accuracy to ensure that its sensitivity to small changes in external impacts be close to the sensitivity of real climate system? A partial answer to this question can be obtained in the framework of the theory of dissipative semidynamic systems.

It is important to emphasize that each modern model of a concrete physical phenomenon is a reflection of contemporary comprehension of the related process. Global climate models which describe a great number of diverse physical processes and their interactions are not an exception in this sense. At present, climate models experience a period of vigorous development spurred by an explosive development of computer technologies. Therefore, we do not consider in detail the parameterization of subscale processes in modern climate models, since these parameterizations are being developed continuously. It is sufficient to mention the parameterization of the formation and transport of clouds and their interaction with radiation. We also avoid presenting the stimulation results for concrete characteristics of contemporary climate, since these results are being continuously improved. Consequently, the focus here is on the established mathematical results of the investigation of climate models and to the scientific directions that are of great importance in the history of climate studies.

Historically, general circulation models of the atmosphere and ocean were a basis for the development of climate models.

The first numerical model of the general circulation of the atmosphere was constructed by N.A. Phyllips. He also has proposed the σ-coordinate system which is very popular nowadays. Systematic investigation of the role of various physical factors in forming the circulation of the atmosphere was started by J. Smagorinsky. The first coupled circulation models of the atmosphere and ocean were constructed by S. Manabe and K. Bryan. Numerical methods for solving the hydrodynamic equations of the atmosphere and ocean were developed due to significant contributions of the international scientific community. A. Arakawa proposed the computational method preserving two quadratic invariants for a two-dimensional incompressible fluid. S.Orszag proposed the method of spectral–mesh transformation, which made the spectral methods widely used in the general circulation models of the atmosphere. G. Marchuk's Siberian school developed a whole class of implicit methods for solving the hydrodynamic equations governing the atmosphere and ocean and, in particular, the splitting method. Qualitative examination of the thermohydrodynamic equations of the atmosphere and ocean was started by the works of the scientists of Russian and French schools.

2. Mathematics for Climate Modeling

2.1. "Ideal" Model of Climate

"Ideal" model of climatic system that generates the observed trajectory of climatic system possesses the following properties:

- The "ideal" model is described by the system of partial differential equations and belongs to the class of dissipative semidynamic systems.
- The "ideal" model has a global attractor.
- The trajectory generated by the "ideal" model lies on its attractor.
- Trajectories on the model's attractor are unstable in the sense of Lyapunov. The dynamics on the attractor is chaotic and ergodic, i.e., the trajectories originated from almost all points of the attractors are everywhere dense on the attractor.

The last assumption is very important, since it opens a possibility to compute the attractor's characteristics possessing only one trajectory (realization).

Assumptions 2 and 3 allow the reformulation of the concept of climate through the characteristics of an attractor of "ideal" model. In particular, if the sense of a "sufficiently long time interval" is understood as an infinite interval, then the climate is understood as an attractor (set) with an ergodic invariant probabilistic measure generated by chaotic dynamics of the "ideal" model specified on this attractor.

The closeness of certain climate model to the "ideal" model should be regarded as the closeness of the above characteristics of the attractors of climate models. In reality, the goodness of the solution is measured in terms of closeness only at certain instants of time or even of their projections on certain subspaces of the original phase space .

Most of climate models used in climate investigations can be reduced to the canonical form.

$$\frac{\partial\varphi}{\partial t} + K(\varphi)\varphi = -S\varphi + f,$$
$$\varphi/_{t=0} = \varphi_0 \qquad (1)$$

with the help of generally nonlinear transformations. Here, φ is the vector function characterizing the state of the system, $\varphi \in \Phi; \Phi$ is the phase space of the system which is regarded as a Hilbert space with the inner product $(\cdot,\cdot)$; f is the external impact which can depend on the solution; S is positive definite operator describing the dissipation in the system:

$$(S\varphi,\varphi) \geq \mu(\varphi,\varphi),\ \mu > 0.$$

$K(\varphi)$ is the skew-symmetric operator linearly depending on the solution:

$$(K(\varphi)\varphi,\varphi) = 0.$$

It is clear that, when $f, S \equiv 0$, there is the quadratic conservation law in the system:

$$\frac{\partial\varphi}{\partial t}(\varphi,\varphi) = 0. \qquad (2)$$

System (1) belongs to the class of dissipative systems, since it has the absorbing set:

$$\|\varphi\| \leq \frac{\|f\|}{\mu}. \qquad (3)$$

Qualitative examination of each concrete climate model means the establishment of the following statements:

- Global theorem of solvability for (1).
- The existence of global attractor.
- The estimation of the attractor's dimension.
- The examination of the attractor's structure and its stability.

In what follows, we discuss these statements when formulating each new climatic model.

2.2 Finite–dimensional Approximation of Differential Climate Models

Equations of climate models are strongly nonlinear, therefore, the only way to examine the behavior of their solutions, attractor structures, and response to small perturbations of external impacts is to construct their finite-dimensional approximations.

Strictly speaking, each concrete finite-dimensional approximation of differential climate model should be regarded as an individual climatic model, since it is necessary to solve the parameterization problem for the subscale processes. The choice of spatial resolution determines spatial scale; the latter implies the solution of an inverse closure problem, which is also an individual problem. Form this standpoint, it is convenient to consider the approximation problem sequentially: first, the system of ordinary differential equations is first approximated with respect to spatial variables and then in time.

Let a spatial finite-dimensional version of system (1) have the form

$$\begin{aligned}&\frac{d\varphi^h}{dt}+K\left(\varphi^h\right)\cdot\varphi^h=\mathrm{f}^h-\mathrm{S_h}\varphi^{\mathrm{h}}\\&\varphi^h/_{\mathrm{t=0}}=\varphi_0^h,\varphi^h\in\Phi_{\mathrm{h}}\end{aligned}\qquad(4)$$

The solvability problem for system (4) is not complicated any more. We require that finite dimensional approximations of the operators $\mathrm{K_h},\mathrm{S_h}$ also possess the properties of skew-symmetry and positive definiteness, respectively:

$$K_h\varphi^h,\ \varphi^h{}_{\Phi_h}=0,\quad\left(\mathrm{S}_h\varphi^h,\varphi^h\right)\geq\ \mu^{\mathrm{h}}\left(\varphi^{\mathrm{h}},\varphi^{\mathrm{h}}\right)\qquad(5)$$

It follows from (5) that, when $\mathrm{f^h},\mathrm{S^h}\equiv 0,$ an analogue of the energy conservation exists in (4):

$$\frac{d}{dt}\left(\varphi^{\mathrm{h}},\varphi^{\mathrm{h}}\right)_{\Phi_{\mathrm{h}}}=0$$

(Here, $(\cdot,\cdot)_{\Phi_{\mathrm{h}}}$ is the inner product in Euclidean space). Moreover, (5) implies the existence of an absorbing set for (4) and the existence of a global attractor in this case for the finite-dimensional system (4) becomes an almost trivial fact. Strictly speaking, when constructing finite-dimensional approximations of system (4), one must prove the theorems about the closeness of attractors of systems (4) and (5). Such closeness must be proved in the Hausdorff metrics:

$$\mathrm{dist_H}\left(\mathrm{A},\mathrm{A}_h\right)=\max\left(\mathrm{dist}\left(\mathrm{A},\mathrm{A}_h\right),\ \mathrm{dist}\left(\mathrm{A}_h,\mathrm{A}\right)\right),$$

where

$$\mathrm{dist}\left(\mathrm{A},\mathrm{B}\right)=\sup_{y\in\mathrm{B}}\inf_{x\in\mathrm{A}}\rho\left(x,y\right),$$

$\rho\left(x,y\right)$ is the metric of the space $\mathrm{R}^{\mathrm{N}*}$, *A* is the finite-dimensional attractor of system(4), and A_h is the attractor of system(5).

Here, a new definition of the approximation of finite-difference schemes arises: the approximation of asymptotic sets. At present, unfortunately, the theorems of closeness of the attractors of differential problems and their finite-dimensional operators have not been proved for all well-known climate models. The establishment of closeness between invariant measures on attractors seems to be a much more complicated problem. Since such theorems are also absent, the construction of finite-dimensional approximations of differential climatic models is a nontrivial task. The approximation of energy intensive large-scale processes and the conception of a group of invariants of climate model in the absence of dissipation and forcing is the basis of such constructions.

Of course, the total energy of system is the basic invariant; it can be reduced to quadratic form by nonlinear transformations. There exists a number of invariants, whose conservation seems to be quite necessary in climatic (finite-dimensional) models. These are the mass conservation law and the conservation law for the angular momentum with respect to the earth's axis of rotation that controls the distribution of winds near the earth's surface.

The atmosphere and ocean can be regarded as quasi-two–dimensional; consequently, "asymptotic" conservation laws are very important, being the conservation laws in the two-dimensional approximation. Two quadratic invariants, the energy and the enstropy, determine, in a two–dimensional fluid, the energy distribution with respect to spatial scales. This is an important climate characteristic associated with the cascade of energy from small scales to larger ones. In fact, it determines the required accuracy of the parameterization of subscale processes.

3. Climatic Models

In the introduction, the climate was defined as an ensemble of states passed by the climatic system over a sufficiently long time interval. In the light of this definition, the climate model is understood as a model that reproduces this ensemble with acceptable accuracy. Consequently, the model that reproduces only certain characteristics of the ensemble cannot be regarded, strictly speaking, as a climatic model. However, following the established tradition in the theory of climate, we regard all the models which reproduce some characteristics of climate as climate models.

Here, the climate models are classified with respect to the dimension of the space in which they are constructed.

3.1 Zero –dimensional Models

The effective temperature can be defined by writing down the radiation balance equation for the whole system. This is possible since the system receives an overwhelming portion of its energy from the sun. Assuming that the climactic system radiates as a gray body, the radiation balance equation is written as

$$\gamma\sigma T^4 = \left(1-\alpha_e\right) I_e, \tag{6}$$

where $I_e = 1/4, I$ is the integral flux of Solar radiation, α_e is the effective albedo of climate system, and γ is the grayness coefficient.

3.2 One–dimensional Models

The class of one-dimensional models includes radiation models, radiation-convection models, and diffusion–convection models. The first two-models are obtained by averaging the heat transport equation over horizontal coordinates:

$$\frac{dT(z)}{dt} = \varepsilon(z) \tag{7}$$

Here, $\varepsilon(z)$ denotes radiative heat fluxes distributed along the vertical coordinate. In the case of radiation-convection models, it also accounts for the vertical mixing by the mechanism of effective convection.

The models obtained by averaging the heat flux equation over the vertical coordinate and longitude belong to another type of models. After certain closure procedure of the eddy heat fluxes, the heat transport equation takes the form:

$$\frac{\partial T}{\partial t} = \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}K(\varphi)\frac{\partial T}{\partial\varphi} + f(T,\varphi), \tag{8}$$

where f is the term describing the heat fluxes. The representation of the effective diffusion coefficient is a key factor in this model. In principle, it must depend on $\left|\frac{\partial T}{\partial\varphi}\right|$, taking thus into account the development of baroclinic vortices in the regions with high temperature gradient.

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Bibliography

Arakawa A. (1966). *Computational design for long-term numerical integration of the equations of fluid motions: two–dimensional incompressible flow. Part I.* J. Comp. Phys., 1 pp. 119-143. [This research, regarded as classic nowadays, described the construction principles of finite-difference schemes for the equations that govern the dynamics of a two-dimensional incompressible fluid; the schemes possessed two quadratic invariants]

Dymnikov V.P., Filatov A.N. (1997). *Mathematics of climate modeling*. Birkhäuser, Boston, 264 p. [In this book, for the first time, results of the investigation were systematically presented concerning models of the circulation of the atmosphere and ocean and based on the methods of a qualitative theory of differential equations]

Journal of climate. (1998). Climate Systems Model Special Issue. v. 11, N 6. [This special issue contains results of the modeling of contemporary climate by a modern NCAR climate model]

Kraichnan R.H. (1959) *Classical fluctuation –relaxation theorem*. Phys. Rev., , 109,pp.1407-1

Lions J.L. , Manley O.P., Temam R.,Wang S. (1997). *Physical interpretation of the Attractor Dimension for the primitive equations of Atmopsheric Circulation*. J.Atm. Sci., v.54, N9, pp.1137-1143.

Manabe S., Bryan K. (1969). *Climate calculations with a combine ocean-atmospheric model*. J. Atm. Sci., 26, pp. 756-789. [This was the first article that presented results of modeling the climate with the help of a coupled model of the general circulation of the atmosphere and ocean]

Marchuk G. I., Dymnikov V.P., and Zalesny V.B. (1987). *Mathematical models in Geophysical Fluid Dynamics and Methods of their Realization* Leningrad, Gidrometeoizdat, 296 pp. (In Russian) [This book describes efficient numerical methods for solving the dynamic equations for the atmosphere and ocean]

Orszag S. (1970). *Transform method for calculation of vector coupled sums. Application to the spectral form of the vorticity equation*. J. Atm. Sci., 27, pp. 890-895. [In this article, the method of spectral–grid transformation was presented that made spectral methods most widely used when solving the equations of the atmosphere’s fluid dynamics]

Philips N.A. (1956). *The general circulation of the atmosphere: a numerical experiment.* Quart. J. Roy. Met.Soc. v.82, N 352, pp.123-164. [In this work, the problem of general circulation of the atmosphere was formulated for the first time on the basis of differential equations of fluid dynamics]

Smagorinsky J. (1963*). General circulation experiment with the primitive equations*. *I. The basic experiment.* Monthly Wea. Rev., v. 93, N 3, pp.99-164. [In this study, the fundamentals of numeric modeling of the general circulation of the atmosphere were formulated for the first time on the basis of primitive dynamic equations for the atmosphere]

Biographical Sketch

Dymnikov Valentin Pavlovich is a, member of the Russian Academy of Sciences and Director of Institute of Numerical Mathematics, Moscow, Russia. He specializes in the filed of geophysical fluid dynamics, modeling of the general circulation of the atmosphere and climate, and numerical mathematics. He is one of the pioneers of the theory of chaotic attractors of climatic models. He published more than 160 scientific works including 11 monographs and text books. He holds the chair of "Mathematical Modeling of Physical Processes" at Moscow Physical and Technological Institute. Academician Dymnikov was awarded by the Fridman Prize of the Russian Academy of Sciences.