EIGENVALUE PROBLEMS: METHODS OF EIGENFUNCTIONS

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Summary

When solving problems of mathematical physics, we often deal with the so-called eigenvalue problems which are represented by a homogeneous linear equation with a parameter. Nontrivial solutions of such an equation (eigenfunctions) play an important

role when determining a solution of an original problem. In some cases special functions, being eigenfunctions of a specific eigenvalue problem, are used. The method of eigenfunctions is one of the most often used methods of mathematical physics. With this method, a solution is represented as the expansion in terms of eigenfunctions of an operator closely related to a problem to be solved. As a rule, this expansion involves orthonormal functions with special weights, namely, Fourier series. The method of eigenfunctions enables us to solve various problems of mathematical physics among which are problems of the theory of electromagnetism, heat conductivity problems, problems of the oscillation theory and acoustics. This method can also be used when solving problems of sustainable development.

1. Introduction

One of the most often used methods of mathematical physics is the method in which a solution is represented in the form of a series in some functions closely related to an original problem which are called eigenfunctions. Physically, in the simplest cases this approach corresponds to superposition of stationary waves.

Some applications of the method of eigenfunctions date back to Euler. Ostrogradskii was the first to develop its general formulation. A rigorous justification of the method is due to Steklov.

The method of eigenfunctions is closely related to the Fourier method, or the method of separation of variables, which is intended for finding a particular solution of a differential equation. When using these methods, we are often concerned with special functions being solutions of an eigenvalue problem. The method of separation of variables was proposed by d'Alembert (1749). In the 18th century it was used by Euler, Bernoulli, and Lagrange for solving the problem of oscillation of a string. Early in the 19th century Fourier developed this method in considerable detail and applied it to the heat conductivity problem. The general formulation of this method is due to Ostrogradskii (1828).

In this chapter the fundamentals of the method of expansion in terms of eigenfunctions are presented and the applications to concrete problems of mathematical physics, among which are problems of the theory of electromagnetism, heat conductivity problems, problems of the oscillation theory and acoustics, are considered.

2. Eigenvalue Problems

Eigenvalue problems often arise when solving problems of mathematical physics. As a rule, an eigenvalue problem is represented by a homogeneous equation with a parameter. The values of the parameter such that the equation has nontrivial solutions are called eigenvalues, and the corresponding solutions are called eigenfunctions.

The simplest eigenvalue problems were considered by Euler. Great attention was paid to these problems in the 19th century when the classical theory of equations of mathematical physics had been established.

2.1. The Formulation of an Eigenvalue Problem and its Physical Meaning

We consider a simple example which reduces to an eigenvalue problem. Assume that a homogeneous string of length l with fixed ends is free of outside forces. Take either of the two ends of the string as the origin and assume that the x axis is directed along the string. A function $u(x;t)$, which describes free small oscillations of the string, satisfies the homogeneous differential equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

and the homogeneous boundary conditions

$$u(0;t) = 0; \quad u(l;t) = 0.$$

The motion of the string is defined not only by the equation and boundary conditions but also by initial conditions.

Consider the simplest motion of the string, namely, stationary waves. The motion such that the shapes of the string at different instants of time are similar to each other is called a stationary wave. A stationary wave is defined by a function of the form

$$u(x;t) = X(x)T(t),$$

where a function $T(t)$, which depends only on time t, is called the law of oscillation and describes the character of motion of individual points of the string, and a function $X(x)$, which depends only on the x coordinate, describes the shape of the string at various instants of time being the same within a factor of $T(t)$.

First, for a string with fixed ends it is obvious that a function $X(x)$ must satisfy the conditions

$$X(0) = 0; X(l) = 0.$$

Besides, $X(x)$ and $T(t)$ must satisfy some equations which follow from the Eq. (1). To obtain these equations, we substitute u, expressed in terms of X and T, into the Eq. (1) that results in

$$X(x)T''(t) = a^2 T(t) X''(x).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side of this equation depends only on t and the right-hand side does not depend on t, both sides are equal to the same constant. We denote this constant by $-\lambda$:

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

Then we have

$$T'' + \lambda a^2 T = 0, \;\; X'' + \lambda X = 0.$$

Thus, we arrive at the simplest eigenvalue problem. It is easy to see that the constant λ can take only the values

$$\lambda_n = \pi^2 n^2 / l^2 \;\; (n = 1, 2, 3, \ldots),$$

and for *the string with fixed ends the shape of stationary waves is defined by*

$$X_n(x) = c \sin \frac{\pi n x}{l}, \quad c = \text{const}.$$

Now we find the functions $T_n(t)$ which correspond to the wave of the shape $X_n(x)$. To this end we substitute the value of λ_n into the equation in T:

$$T''_n + \frac{a^2\pi^2}{l^2} n^2 T_n = 0.$$

The general solution of this equation has the form

$$T_n(t) = B_n \sin \frac{\pi a n}{l} t + C_n \cos \frac{\pi a n}{l} t = A_n \sin\left(\frac{\pi a n}{l} t + \varphi_n\right),$$

where B_n and C_n or A_n and φ_n are arbitrary constants.

With the use of X_n and T_n, we can write the final expression for all admissible stationary waves:

$$u_n(x;t) = A_n \sin\left(\frac{\pi a n}{l} t + \varphi_n\right) \sin \frac{\pi n x}{l},$$

where n=1, 2, 3, ….

Thus, the n-th *stationary wave describes the motion of the string such that each point of* *the string executes harmonic oscillations with frequency* $\frac{\pi an}{l}$ being the same for all points . The amplitudes of these oscillations vary from point to point and are equal to $A_n \left|\sin\frac{\pi nx}{l}\right|$ (A_n is arbitrary).

Since free oscillations of the string are uniquely defined by its initial shape $u|_{t=0}$ and by initial velocities $\frac{\partial u}{\partial t}|_{t=0}$ of its points, it is obvious that a stationary wave arises if and only if the initial deflection and the initial velocity have the form

$$u\big|_{t=0} = D\sin\frac{\pi nx}{l}; \quad \frac{\partial u}{\partial t}\big|_{t=0} = E\sin\frac{\pi nx}{l}, \quad D, E = \text{const.}$$

This stationary wave is defined by the equation

$$u(x;t) = \left(\frac{l}{\pi an}E\sin\frac{\pi an}{l}t + D\cos\frac{\pi an}{l}t\right)\sin\frac{\pi nx}{l}.$$

The quantities λ_n are called eigenvalues and the functions $X_n(x)$ are called eigenfunctions.

We introduce a general definition of eigenvalues and eigenfunctions. Let L be a linear operator with a domain $D(L).$ We consider a homogeneous linear equation

$$Lu = \lambda u, \qquad (2)$$

where λ is a complex parameter. This equation has the trivial solution for all $\lambda.$ For some λ this equation may have nontrivial solutions $D(L).$ A complex value of λ such that the Eq. (2) has nontrivial solutions which belong to $D(L)$ is called *an eigenvalue* of the operator L, and the solutions themselves are called *eigenfunctions* corresponding to this eigenvalue. The total number $r(1 \le r \le \infty)$ of linearly independent eigenfunctions, corresponding to an eigenvalue $\lambda,$ is called the *multiplicity* of this eigenvalue; if $r = 1,$ then λ is called a *simple* eigenvalue.

If the multiplicity r of an eigenvalue λ of an operator L is finite and $u_1, u_2, \ldots, u_r$ are corresponding linearly independent eigenfunctions, then any linear combination

$$u_0 = c_1u_1 + c_2u_2 + \ldots + c_ru_r$$

is also an eigenfunction corresponding to this eigenvalue, and this formula gives the general solution of the Eq. (2). Hence if *the equation*

$$Lu = \lambda u + f$$

has a solution, then its general solution is defined by the formula

$$u = u^* + \sum_{k=1}^{r} c_k u_k,$$

where u^* *is a particular solution and* c_k, $k = 1, 2, \ldots, r$, *are arbitrary constants.*

Eigenvalues and eigenfunctions often have clearly defined physical meaning: in the example considered above the eigenvalues λ_n define the frequency of harmonic oscillations of the string, and the eigenfunctions X_n define amplitudes of oscillations.

2.2. Eigenvalue problems for differential operators

We consider a more general case of a mixed problem for a homogeneous differential equation with homogeneous boundary conditions.

Let Ω be a bounded domain in a one-, two-, or three-dimensional space. We denote an arbitrary point of the domain Ω by P and a function of coordinates of this point by $u(P)$. We consider a linear differential operator of the function u of the form

$$L[u] \equiv \operatorname{div} p \operatorname{grad} u - qu,$$

where functions $p(P)$ and $q(P)$ are continuous inside Ω and on its boundary $\partial\Omega$. Moreover, we assume that $p(P) > 0$ inside the domain Ω and on the boundary.

In the one-dimensional case Ω is an interval (a, b) on the x axis. In this case the operators 'grad' and 'div' mean $\frac{d}{dx}$, hence,

$$L[u] = \frac{d}{dx}\left[p(x)\frac{du}{dx} \right] - q(x)u \equiv p(x)\frac{d^2u}{dx^2} + p'(x)\frac{du}{dx} - q(x)u.$$

On the boundary $\partial\Omega$ of the domain Ω we consider homogeneous boundary conditions of the form

$$\Lambda[u] = 0,$$

where

$$\Lambda[u] \equiv p\frac{\partial u}{\partial n} - \gamma u,$$

or

$$\Lambda[u] \equiv u.$$

In the former case we deal with so-called *boundary conditions of the third kind* (or, if

$\gamma = 0$, of the second kind), in the latter case we have *boundary conditions of the first kind*. Here γ is a continuous nonnegative function defined on $\partial\Omega$, and n is the direction of an inner normal to $\partial\Omega$.

In the one-dimensional case the boundary of a domain consists of two endpoints a and b of a segment and by the derivative $\frac{\partial}{\partial n}$ is meant $\frac{d}{dx}$ at the point a and $-\frac{d}{dx}$ at the point b. Then to define the function γ it is sufficient to specify two nonnegative numbers γ_a and γ_b, and the operator $\Lambda[u]$ is defined by

$$\Lambda_a[u] = p(a)\frac{du(a)}{dx} - \gamma_a u(a), \quad \Lambda_b[u] = -p(b)\frac{du(b)}{dx} - \gamma_b u(b).$$

Sometimes boundary conditions of other kind, namely, periodic conditions, are considered. For example, in the one-dimensional case these conditions are defined by the equalities $u(a) = u(b)$, $p(a)u'(a) = p(b)u'(b)$. These boundary conditions are homogeneous as well, but a feature of these conditions is that both points a and b enter into each equality.
Consider the following problem

$$L[u] + \lambda\rho u = 0 \text{ in } \Omega, \qquad (3)$$

$$\Lambda[u] = 0 \text{ on } \partial\Omega, \qquad (4)$$

where $\rho = \rho(P)$ is a nonnegative continuous function defined in the domain Ω. This function is called a *weight function* of the problem or a *weight*. As in the case of a string, for all values of λ there exist solutions u satisfying the boundary conditions. The values of the parameter λ, such that the Eq. (3) has nontrivial solutions satisfying the boundary conditions (4), are eigenvalues, and the corresponding solutions u are eigenfunctions of the operator L. If a number of eigenfunctions is so "large" that any function defined in the domain Ω (satisfying some natural smoothness requirement) can be expanded into a series in terms of these eigenfunctions, then we can seek a solution of a nonhomogeneous problem as a series in terms of corresponding eigenfunctions.

In the following subsection we consider some well-known properties of the eigenvalue problem (3)–(4).

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