NUMERICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMIC SYSTEMS

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Summary

Approximate methods of solution of Cauchy problem for systems of ordinary differential equations, including delay differential equations, are described. Principal methods of local and global error estimation as well as the inequalities for accuracy control of computations and stability control of numerical schemes are considered. Realization of variable step algorithms and algorithms for non-uniform schemes (explicit and L-stable) with automatic selection of numerical scheme by inequality for stability control is briefly discussed. Examples of particular numerical methods are given.

1. Introduction

Early analysis of ordinary differential Eqs. (ODE) can be traced towards the end of the 17th century in connection with the study of problems of mechanics and certain geometric problems. The unknown in these equations is a function of one independent variable, and the equations include not only the unknown function but its derivatives of diverse orders as well.

ODEs are widely used in mechanics, astronomy, physics, biology, chemistry etc. It is explained by the fact that such equations often quantitatively are based on physical laws, which describe some phenomena. Schematic design of electronic schemes, modeling of kinetics of chemical reactions and computation of dynamics of mechanical systems is a far from complete list of the problems described by ODE. A large class of problems arises from the approximation of time-dependent problems by finite elements or finite differences. In turn, applied problems serve as a source for new problem statement in the theory of differential equations. For instance, in this way the optimal control mathematical theory had been developed.

Elementary ODEs occur when finding for the integral of a given continuous function $f(t)$, since this is a problem of finding unknown function $y(t)$, which satisfies the equation $y' = f(t)$. For proof of solvability of this equation the Riemann integral theory had been developed. A natural generalization of this equation is a system of ODEs of the first order solved for derivative

$$y' = f(t, y), \tag{1a}$$

where $f : R \times R^N \to R^N$ is known continuous vector-function, and R^N is N-dimensional linear real space.

The relation (1a) is an abridged form of writing the system of equations $y_i' = f_i(t, y_1, \ldots, y_N), 1 \le i \le N$. The solution of (1a) on an interval $[a,b]$ is a function $\varphi : [a,b] \to R^N$, which turns (1a) into identity on $[a,b] : \varphi'(t) \equiv f(t, \varphi(t)), t \in [a,b]$. In general situation Eq. (1a) has infinite set of solutions (N-parametric family of solutions). To choose one of them, a supplementary condition is required, for example, requirement for the solution to take on a given value y_0 at the given point t_0:

$$y(t_0) = y_0. \tag{1b}$$

The problem of finding the solution of system of ODEs (1a), which satisfies the initial condition (1b), is called Cauchy problem or initial-value problem. In what follows, as problem (1) the problem (1a), (1b) is understood.

In order that the problem (1) would have unique solution, it is necessary to impose supplementary requirements on the function f. It is said that function $f(t, y)$ satisfies Lipshitz condition in the second argument, if there exists such constant L that for all $t \in R$ and $x, y \in R^N$ the inequality $\| f(t,x) - f(t,y) \| \le L \| x - y \|$ is valid, where $\| \cdot \|$ is certain norm in R^N. The number L is called Lipshitz constant.

If function f is continuous in the first argument and satisfies Lipshitz condition in the second, then problem (1) has unique solution on any interval of the form $[t_0, t_k], t_k \ge t_0$. A simple sufficient condition for Lipshitz condition is the following one. If function f is differentiable in the second argument and its derivative is uniformly bounded by some constant $L : \| \partial f(t,y) / \partial y \| \le L$ for all $(t, y) \in R \times R^N$, then the function satisfies Lipshitz condition with constant L.

2. Dynamic Systems

Initially, as dynamic system a mechanical system with finite number of degrees of freedom was understood. State of such system is characterized by its disposition (configuration), and law of motion determines the rate of change of the state of the system. In the simplest case the state is characterized by values $y_1,\cdots,y_N$, which can take on arbitrary real values. Two different collections $y_1,\cdots,y_N$ and $\tilde{y}_1,\cdots,\tilde{y}_N$ correspond to different states, and proximity of y_i and $\tilde{y}_i, 1 \le i \le N$ means closeness of corresponding states of the system. In this case the law of motion is written down in the form of autonomous ODE system. Considering the values $y_1,\cdots,y_N$ as coordinates of point y in N -dimensional space, one can geometrically represent the states of dynamic system by means of this point. Such space is called phase space or state space, and the change of state in time is called phase trajectory or state trajectory. Later on, the notion of dynamic system got wider interpretation and meant arbitrary physical system describable by autonomous system of ODE. Dynamic system is talked about when qualitative behavior of all trajectories is considered in the whole phase space (global theory) or in some its part (local theory). In the theory of dynamic systems a special attention is paid to behavior of phase trajectories under infinitely increase of time. From particular trajectories usually those are of interest, whose properties can influence greatly the qualitative pattern, even if local one. Since non-autonomous system can be reduced to autonomous by introducing supplementary variable, these two problems are not discriminated below, if the difference is not of principal case.

3. Analytical methods

A large part of theory of ODE is devoted to the study of solutions, which are not known exactly. This is so-called qualitative theory of differential equations. It includes stability theory, which enables us to indicate stability properties of the solutions from properties of the equation without knowing the solution. For example, in the control theory there exists a need to ascertain stability of solution with regard to small perturbation of initial values in the whole infinite interval $t \ge t_0$. Solution, which varies little in infinite interval $[t_0,\infty]$ under small perturbation of initial values, is called stable according to Liapunov. However, in many particular problems it is often necessary to find solution at every point. Initially the efforts were concentrated on integration of equations by quadrature methods, i.e. on obtaining formulae expressing (explicitly, implicitly or in parametric form) the dependence of solution on t via elementary functions or integrals of the latter. But in the middle of 19th century the first examples of ODE non-integrable by quadrature methods had been indicated. It turned out that solution in form of formulae can be found for small number of classes of equations (Bernoulli equation, differential equation in total differentials, linear differential equation with constant coefficients). Even the simplest non-linear equation of the first order $y' = y^2 + t^2$ cannot be solved by quadrature methods. Therefore methods were required for computation of approximate solutions of differential equations.

Historically, the first method of solution of ODEs used by its author Newton was the method of series expansion. The desired solution is expanded into a series (for instance, Taylor series) with unknown coefficients. This series is substituted in (1a), and from the obtained equation the coefficients are determined. If function f is analytic, that is, can be expanded into power series in t and y: $f(t,y) = f_{00} + f_{01}y + f_{10}t + f_{20}t^2 + f_{11}ty + f_{02}y^2 + \cdots$, then solution $y(t)$ of problem (1) is analytic as well and can be represented in the form $y(t) = y_0 + y_1 t + y_2 t^2 + \cdots$ with unknown coefficients $y_0, y_1, y_2, \cdots$. After substitution of the obtained expressions in (1) and equalization of coefficients at equal powers of t an infinite system of equations $y_1 = f_{00} + f_{01}y_0 + f_{02}y_0^2 + \cdots,\ 2y_2 = f_{10} + f_{01}y_1 + f_{11}y_0 + 2f_{02}y_0y_1 + \cdots,\ \ldots$ is obtained. From the first equation y_1 is found, from the second – y_2 and so on. These methods require a large amount of tedious work. For finding coefficients of Taylor series it is necessary to compute derivatives of high orders of the function $f(t,y)$.

To find approximate solution, the method of successive approximations can be used. Problem (1) is equivalent to integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds,\ t \in [t_0, t_k]. \qquad (2)$$

Applying to the obtained equation the method of simple iteration, starting, for instance, from function $y^{(0)}(t) = y_0$, one obtains recurrently determined sequence of functions

$$y^{(n)}(t) = y_0 + \int_{t_0}^{t} f(s, y^{(n-1)}(s))ds,\ n = 1,2,\cdots. \qquad (3)$$

The sequence of functions $y^n(t)$ uniformly converges to the solution of Eq. (1), estimation of the rate of convergence being known. Since this method requires large amount of computational work, it plays mainly theoretical role – for example, it is useful when proving Cauchy-Picard theorem or when proving statements on differentiability of solutions with respect to parameter or initial data.

In applications, asymptotic methods of approximate solution of ODE are often used. Instead of (1) a simpler integrable problem is solved by quadrature methods, whose solutions approximate solutions of the original one.

Asymptotic methods are based on distinguishing in the equation principal terms and the terms relatively small.

As an example the method of small parameter for equation $y' = f(t, y; \mu)$ can be adduced. Asymptotic methods are used both for obtaining analytic expressions which approximate solution and for study of qualitative behavior of the solution.

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Biographical Sketch

Evgenii A. Novikov was born in Voronezh, Russia. He completed his Diploma in Applied mathematics at the Voronezh State University, Voronezh, Russia in 1978. In 1982 he took the Russian degree of Candidate in Physics and Mathematics at the Computer Center of the Russian Academy of Sciences, Novosibirsk. In 1992 E.A.Novikov was awarded the Russian degree of Doctor in Physics and Mathematics from Computer Center of the Russian Academy of Sciences, Novosibirsk with the thesis "One-step non-iterative methods of solution of stiff systems". In 1993 he took Professor Diploma at the Chair "Mathematical support of discrete devices and systems". From 1978 to 1985 he is as a research worker at the Computing center SB RAS in Novosibirsk. Since 1985 E.A. Novikov is Head-Scientist at the Computer Center of the Russian Academy of Sciences in Krasnoyarsk, Russia (now renamed the Institute of Computational Modeling of the Russian Academy of Sciences). E.A. Novikov is a known specialist in the field of computational mathematics. He is author of one monograph and over 120 scientific papers, devoted to numeric methods of solution of ordinary differential equations, and their applications.