COMPUTATIONAL METHODS IN ELASTICITY

Michel SALAUN
Départment de Mathématiques et Informatique, ENSICA, Toulouse, France

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Contents

1. Introduction

The aim of this chapter is to introduce the reader to basic aspects of numerical methods in linear elasticity. This subject is so vast that it is not possible to cover it exhaustively here. So, we have chosen to focus on some general aspects and to point out some particular difficulties. The outline of the chapter is therefore as follows. After a brief review of continuum mechanics, we state the problem of three-dimensional linearized elasticity, first in static and then in dynamic forms. For these two problems, the main theoretical results are given, and in the elastodynamics case, we make a survey of the two algorithms which are commonly used to solve this problem, and we point out their main advantages and drawbacks. The last part of the chapter is devoted to a very common particular case in elasticity: thin structures (beams, plates and shells). Here, it is possible to “simplify” the formulations in deriving a one or two-dimensional problem from the original three-dimensional problem. The process is illustrated with plates and some examples of the numerical difficulties are discussed.

Finally, for a more general and complete overview of the subject, we refer the reader to the books listed in bibliography.

2 Basic Aspects of Continuum Mechanics

2.1 Strain Tensor

In space $\mathbb{R}^3$, referred to as an orthonormal system of coordinates, say $(O;x_1,x_2,x_3)$, we consider a solid body which occupies, in its reference state, the bounded open sets Ω and, after deformation, the open set Ω'. We denote by $u=u(x_1,x_2,x_3)\equiv u(x)$ the three-dimensional displacement of any point M of Ω with $OM=(x_1,x_2,x_3)$. We denote by M' the image of M through the mapping u. It means that: $M\,M'=u(x)$. So, if we let M_1 and M_2 be two points of Ω, we have:

$$M_1'M_2' = M_1M_2 + u(x^{(2)}) - u(x^{(1)}) .$$

If we suppose that u is a differentiable function, a first order Taylor expansion gives:

$$M_1'M_2' = M_1M_2 + \frac{\partial u}{\partial M}(x^{(1)})\cdot M_1M_2 + \|M_1M_2\|O(M_1M_2) \quad ,$$

where $\frac{\partial u}{\partial M}$ is the tensor whose components are $\frac{\partial u_i}{\partial x_j}$. Moreover, $O(M_1M_2)$ is a vector function which tends to 0 as $\|M_1M_2\| \to 0$ (following this section, $O(M_1M_2)$ will denote various functions which have the same property). Introducing the tensor $F = I + \frac{\partial u}{\partial M}$, we obtain:

$$M_1'M_2' = F\,.M_1M_2 + \|M_1M_2\|O(M_1M_2) \quad ,$$

(I is the identity operator). Then, the relative length variation of segment M_1M_2 of the solid Ω, is given by:

$$\delta(M_1M_2) = \frac{\|M_1'M_2'\| - \|M_1M_2\|}{\|M_1M_2\|} \quad ,$$

and it is easy to see that:

$$\delta(M_1M_2) = \left(\frac{M_1M_2^T \, . \, F^T \, . \, F. \, M_1M_2}{M_1M_2^T \, . \, M_1M_2} \right)^{1/2} - 1 + O(M_1M_2),$$

where F^T is the transpose of F. Then, we introduce the **total strain tensor**:

$$\tilde{\gamma}(u) = F^T \, . \, F - I = \frac{1}{2}\left(\frac{\partial u}{\partial M} + \frac{\partial u^T}{\partial M} + \frac{\partial u^T}{\partial M}\frac{\partial u}{\partial M} \right),$$

which is symmetric but does not depend linearly on the displacement u. And then, **the total strain tensor measures the relative length variation** because we have:

$$\delta(M_1M_2) = \left(1 + 2\frac{M_1M_2^T \, . \, \tilde{\gamma}(u) \, . \, M_1M_2}{M_1M_2^T \, . \, M_1M_2} \right)^{1/2} - 1 + O(M_1M_2),$$

and using $\sqrt{1+\varepsilon} \sim 1 + \frac{\varepsilon}{2}$ when ε is small, we deduce that:

$$\delta(M_1M_2) \simeq \frac{M_1M_2^T \, . \, \tilde{\gamma}(u) \, . \, M_1M_2}{M_1M_2^T \, . \, M_1M_2}.$$

2.2 Linearized Strain Tensor and Small Displacements Hypothesis

The small displacements hypothesis consists in assuming that the tensor $\frac{\partial u}{\partial M}$ is negligible relative to the identity tensor, in the expression for the tensor F. So the total strain tensor is approximated by the **linearized strain tensor**:

$$\gamma(u) = \frac{1}{2}\left(\frac{\partial u}{\partial M} + \frac{\partial u}{\partial M}^T \right),$$

which is also symmetric but now depends linearly on the displacement u. The components of the tensor $\gamma(u)$ are:

$$\gamma_{ij}(u) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right),$$

where Latin indices $i, j, k, l....$ take on values 1,2 and 3.

An important problem is to characterize displacements whose strain tensor is null. For example, in the case of the total strain tensor, $\tilde{\gamma}(u)$ =0 means that, for each segment M_1M_2, we have $\delta(M_1M_2)$ =0. The length of the segment is constant and the displacement u is an isometry. Let us now examine the case of the linearized strain tensor.

Theorem 2.1. *The displacement fields* u, *such that* $\gamma(u)$ =0, *are characterized by:*

$$u(x) = A + \omega \times OM$$

where A *and* ω *are constant vectors in* $\mathbb{R}^3$. *The first one is a* ***translation*** *movement while the second is an infinitesimal* ***rotation****. Such a movement is called a* ***rigid-body motion****.*

Let u be a displacement such that $\gamma(u)$ =0. Then, for all *i* and *j*, we have:

$$\gamma_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) = 0.$$

A simple calculation leads to the following identity for all vector field υ:

$$\partial_{ij}\upsilon_k = \partial_j \gamma_{ik}(\upsilon) + \partial_i \gamma_{jk}(\upsilon) - \partial_k \gamma_{ij}(\upsilon),$$

which implies that displacement u is such that: $\partial_{ij} u_k = 0$ for all $i, j,$ and *k*. Consequently, u is a first order polynomial function: $u_k(x_1, x_2, x_3) = a_k + b_{ki} x_i$, where a_k and b_{ki} are constants. Moreover, we adopt the summation convention on repeated indices. In a vector form, this equation can be written as: $u(x) = A + B\,.\,OM$, where A is the vector of components a_k and B the matrix of components b_{ij}. Then, putting the expression in $\gamma_{ij}(u) = 0$, we deduce that: $b_{ij} + b_{ji} = 0$. Matrix B is antisymmetric so can be represented by a vector product: there exists a vector, say ω, such that: $B\,.\,OM = \omega \times OM$ (vector product), which gives the stated result.

Remark 2.2. It is important to note that a *vector product is not really a rotation*. But when the norm of the vector ω is small, which is precisely the frame of small displacements hypothesis, this approximation is justified. Geometrically, it is like approximating the tangent of an angle by the angle itself.

2.3 Stress Tensor

Let us go back to the solid body which occupies the bounded open set Ω. Now, forces are applied on it. Physically speaking, there only exist two types of forces, even if it is sometimes convenient to use concentrated forces, as point or line ones.

-The first ones are **volume forces**, for example self weight, electric forces or inertia forces. They are applied in the volume of Ω. In the following, their density will be denoted by f.

-The second are **surface forces**, for example pressure or aerodynamic forces. They are applied on the boundary Γ of Ω. In the following, their density would be denoted by g.

Let us now consider a thin short cylinder inside Ω, whose axis is parallel to a given vector n. Let M be the center of this cylinder. The circular section, say dS, containing M, divides the cylinder into two parts, say *I* and *II*, the cylinder being oriented from *I* towards *II* by n. When the body is loaded, each "half-cylinder" exerts force on the other. By definition, the **stress vector** at point M along the direction n is the vector C, such that the load applied by the part *II* of the cylinder on *I*, is equal to $C\ dS$. Let us note that the component of C along n is called the **normal stress**, while the tangential part is called the **shear stress**.

From a practical point of view, one could be interested in having the stress vector at M along any direction in $\mathbb{R}^3$. That is why one introduces the (Cauchy) **stress tensor** in the following way. The space being referred to an orthonormal system of coordinates, we can define the stress vector at M along the three directions $(O;x_i)$, say σ_i. Then, we build the tensor, say σ, with three lines and three columns, where column i contains the components of σ_i. The two main properties of this tensor are the following. The stress vector along any direction n is the vector $C = \sigma . n$. This property is obtained by writing the equilibrium of the forces applied on a small tetrahedron containing point M. Then, using the equilibrium of the momentum applied on a small parallelepiped around M, we can show that **the stress tensor is symmetric**.

Remark 2.3. A stress is a surface force. In the international system of units, it is expressed in **Pascals**. One Pascal corresponds to the pressure due to a force of one Newton acting on a surface of one square meter. In practice, the mega-Pascal, denoted by MPa and equal to 10^6 Pa, is very often used.

2.4 Generalized Hooke's Law

We adopt the small displacements hypothesis, and assume that the initial configuration of the body is its natural state. Then, the stress tensor is defined in terms of the (linearized) strain tensor by the generalized Hooke's law (summation convention on repeated indices):

$$\sigma_{ij} = R_{ijkl} \quad \gamma_{kl}(u),$$

where R_{ijkl} are given functions of the position x called elasticity coefficients. Tensor R is called the **stiffness tensor**. When the R_{ijkl}'s are constant, the body is called homogeneous . Moreover, the stiffness tensor is assumed to satisfy the following symmetry properties:

$$R_{ijkl} = R_{jikl} = R_{ijlk} = R_{klij} .$$

The two first symmetry properties are due to the corresponding symmetries on stress and strain tensors. The last one is due to thermodynamics considerations. Nevertheless, tensor R depends on 21 different coefficients, which are obtained experimentally.

In the particular case of a **homogeneous and isotropic material**, tensor R only depends on 2 coefficients: the Lamé coefficients, denoted by λ and μ, or, equivalently, Young's modulus E and Poisson's ratio ν. Then, Hooke's law is written:

$$\sigma_{ij} = \frac{\mathrm{E}}{1+\nu}\left(\gamma_{ij}(u) + \frac{\nu}{1-2\nu}\gamma_{ll}(u)\,\delta_{ij}\right).$$

In this expression, $\gamma_{ll}(u) \equiv \gamma_{11}(u) + \gamma_{22}(u) + \gamma_{33}(\mu)$, and δ_{ij} is the Kronecker symbol, *i*.e. $\delta_{ij} = 1$, for $i = j$ and δ_{ij} = 0, for $i \neq j$.

Remark 2.4. We have the following relations between E and ν, on the one hand, and λ and μ on the other:

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \qquad \nu = \frac{\lambda}{2(\mu+\lambda)}$$

and conversely:

$$\mu = \frac{\mathrm{E}}{2(1+\nu)} \qquad \lambda = \frac{\mathrm{E}\nu}{(1+\nu)(1-2\nu)} .$$

In the international system of units, E, λ and μ are expressed in Pascals. ν is non-dimensional .

3. The Three-Dimensional Linearized Elasticity

3.1 Variational Formulation

Let Ω be a bounded open subset of $\mathbb{R}^3$ and Γ, its boundary. We split the boundary Γ into two parts: $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\Gamma_0 \cap \Gamma_1 = \varnothing$. Then, the solid body, which occupies the domain Ω, is assumed to be fixed on Γ_0. Moreover, the solid is subjected to a volume force field, of density f, and to surface forces of density g along Γ_1.

Let us introduce the space of admissible displacements:

$$V = \{\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) / \upsilon_i \in H^1(\Omega); \upsilon_i = 0 \text{ on } \Gamma_0\}. \qquad (3.1)$$

Then the stress tensor is result of application of what is called, in solid mechanics literature, the **principle of virtual work:**

$$\forall \upsilon \in V, \qquad \int_\Omega \sigma_{ij} \gamma_{ij}(\upsilon)\, dx = l(\upsilon).$$

The linear form l corresponds to the loading applied to the structure, and is defined by:

$$l(\upsilon) = \int_\Omega f_i \;\upsilon_i\; dx + \int_{\Gamma_1} g_i \;\upsilon_i\; d\Gamma(x). \qquad (3.2)$$

Then, introducing Hooke's law, we obtain the variational formulation of the three-dimensional linearized elasticity:

$$\begin{cases} \text{Find } u \in V \text{ such that for all } \upsilon \in V \quad : \\ a(u,\upsilon) \equiv \int_\Omega R_{ijkl}\; \gamma_{kl}(u)\; \gamma_{ij}(\upsilon)\; dx = l(\upsilon). \end{cases} \qquad (3.3)$$

Because of the symmetry of the stiffness tensor: $R_{ijkl} = R_{klij}$, the bilinear form a is symmetric : $a(u,\upsilon) = a(\upsilon,u)$. Now, we can state the main result of this section.

Theorem 3.1. *Let* Ω *be a bounded connected open subset of* $\mathbb{R}^3$ *with a Lipschitz continuous boundary, and let* Γ_0 *be a measurable subset of* Γ, *whose measure (with respect to the superficial measure* $d\Gamma(x)$*) is* ***strictly positive****. Let us assume that the stiffness tensor satisfies the two following properties*:

(1) $R_{ijkl} \in L^\infty(\Omega)$ *for all indices i, j , k , and l.*

(2) *Ellipticity : there exists a strictly positive constant* C *such that*:

for all symmetric tensor τ, $\qquad R_{ijkl}\;\tau_{ij}\;\tau_{kl} \;\geq\; C\;\tau_{ij}\;\tau_{ij}$.

Then, there exists a unique $u \in V$ *solution of the variational problem (3.2)-(3.3).*

The proof is based on Lax-Milgram theorem and detailed in *Variational Statement of Problems. Variational Methods* (*cf.* also [2] or [3]). In this section, we just focus on some points. Property (1) ensures that the bilinear form a is continuous on V, while property (2) is used to prove the ellipticity of a. But we need also an additional result called Korn's inequality: there exists a strictly positive constant, say C, depending on Ω, such that, for all $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in [H^1(\Omega)]^3$, we have:

$$\| \upsilon \|_{1,\Omega} \le C \left\{ \sum_{i,j=1}^{3} \| \gamma_{ij}(\upsilon) \|_{0,\Omega}^{2} + \sum_{i=1}^{3} \| \upsilon_i \|_{0,\Omega}^{2} \right\}^{1/2} .$$

Then, it is possible to prove the following kind of Poincaré-Friedrichs inequality:

Theorem 3.2. *If* Ω *is a bounded connected open subject of* $\mathbb{R}^3$ *with a Lipschitz continuous boundary, and if the measure of* Γ_0 *is strictly positive, the mapping:*

$$\upsilon \mapsto \left\{ \sum_{i,j=1}^{3} \| \gamma_{ij}(\upsilon) \|_{0,\Omega}^{2} \right\}^{1/2} \qquad (3.4)$$

is a norm on V *which is equivalent to the usual norm* $\| \,.\, \|_{1,\Omega}$.

We will not prove this result, but only check that the mapping (3.4) is a norm on V . Let us consider a displacement field υ such that : $\left\{\sum_{i,j=1}^{3} \| \gamma_{ij}(\upsilon) \|_{0,\Omega}^{2}\right\}^{1/2}$ = 0. Then, the linearized strain tensor associated with υ is null. Consequently, υ is a rigid-body motion (*cf.* Theorem 2.1): $\upsilon(x) = A + \omega \times OM$. As Γ_0 has a strictly positive measure, there exist at least *three points which are not on the same line*. So we deduce that vectors A and ω are necessarily null, and displacement υ also.

Remark 3.3. This result is very important in practice. It shows that **the boundary condition must eliminate the rigid-body motions**, *i.e.* translations and rotations, if we want the problem to be well posed. One has to keep this is mind when introducing the boundary conditions in a computational software.

To conclude this section, let us just give two immediate consequences of these results (the details can be found also in *Variational Statement of Problems. Variational Methods*).

(1) The displacement u, solution of (3.2)-(3.3), continuously depends on the data. There exists a strictly positive constant C such that for all solution u:

$$\| u \|_{1,\Omega} \le C \{ \| f \|_{0,\Omega} + \| g \|_{0,\Gamma_1} \}.$$

(2) As the bilinear form a, given in (3.2)-(3.3), is symmetric and positive definite, the solution u realizes the minimum of the strain energy, say J, of the structure:

$$J(\upsilon) \equiv \frac{1}{2} a(\upsilon,\upsilon) - l(\upsilon) = \frac{1}{2} \int_\Omega R_{ijkl} \ \gamma_{kl}(\upsilon) \ \gamma_{ij}(\upsilon) dx - \int_\Omega f_i \ \upsilon_i \ dx - \int_{\Gamma_1} g_i \ \upsilon_i \ d\Gamma(x) .$$

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Biographical Sketch

Michel Salaun obtained a diploma of engineer at the Ecole Centrale de Paris in 1983. Then he completed his studies by a thesis under the direction of Professor Philippe Destuynder and got his Habilitation in applied mathematics in 1998. From 1991, he gave lectures in scientific computing and numerical simulation in structural mechanics, in the "Conservatoire National des Arts et Métiers". Now he teaches also mathematics and scientific computing in ENSICA, a French high school devoted to aeronautics.
His major field of research activity is the use of numerical methods in several domains of mechanics. First he developed a mixed finite element for Koiter's shell model, in the framework of linear elasticity, in order to avoid numerical locking. Then this formulation was used for the development of an Eulerian approach for large displacements of thin shells, with Inigo Arregui, and for the modeling of the unilateral contact between spot-welded shells, with Isabelle Vautier. Second, François Dubois and Stéphanie Salmon, he worked on the stream function-vorticity formulation of the Stokes problem and introduced harmonic discretization of the vorticity along the edge, which improved the theoretical and numerical results obtained for this problem. The method is now being extended to a vorticity-vorticity-pressure formulation for incompressible flows.