DUALITY THEORY

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Contents

Summary

The purpose of this chapter is to present the duality theory in mathematical programming. The mathematical setup of duality depends on the actual problem under study, which for example may be an integer programming problem or a convex programming problem. The main intention is to introduce a unifying framework clearly exhibiting the basic involutory property of duality. It is then demonstrated how to derive a duality theory for some of the most important classes of mathematical programming problems. Emphasis is put on the description of the interrelationships among the various theories of duality. Detailed mathematical derivations, most of which are easily available in the textbooks mentioned, have not been included.

1. Introduction

Duality is an important concept in many areas of mathematics and its neighboring disciplines. *Encyclopedia Britannica* explains the concept as follows: "In mathematics, principle whereby one true statement can be obtained from another by merely interchanging two words." Here we shall consider duality in the context of optimization and the two *words* to be interchanged are going to be the terms: *maximum* and *minimum*.

First some notation: Introduce two arbitrary nonempty sets S and T and a function $K(x, y) : S \times T \to \mathbb{R}$. Consider the following *primal* problem.

$$z_0 = \max_{x \in S} \min_{y \in T} K(x, y) \quad . \tag{1}$$

Let $x_0 \in S$. If z_0 is finite and $z_0 = \min_{y\in T} K(x_0, y)$ then x_0 is said to be an *optimal solution* of the problem. (For simplicity of notation we assume that for fixed x the inner minimization problem of (1) is obtained for an argument of $y \in T$ unless it is unbounded below).

Thus, the primal problem is considered as an optimization problem with respect to (w.r.t.) the variable x. We shall next introduce the *dual* problem as an optimization problem w.r.t. y.

$$w_0 = \min_{y\in T} \max_{x\in S} K(x, y)\,. \qquad (2)$$

Let $y_0 \in T$. If w_0 is finite and $w_0 = \max_{x\in S} K(x, y_0)$ then y_0 is said to be an optimal solution of the problem. (Again, for simplicity of notation we assume that the inner maximum w.r.t. x is obtained unless the value is unbounded above).

Note that the dual problem is equivalent to

$$-\max_{y\in T} \min_{x\in S} -K(x, y)\,,$$

which has the form of a primal problem. By the above definition its dual problem is

$$-\min_{x\in S} \max_{y\in T} -K(x, y)\,,$$

which is equivalent to (1). This exhibits the nice property of *involution* which says that *the dual of a dual problem is equal to the primal problem*. We may thus speak of a pair of mutually dual problems, in accordance with the above quotation from *Encyclopedia Britannica*.

The entire construction may be interpreted in the framework of the so-called zero-sum game with two players, player 1 and player 2. Player 1 selects a strategy x among a possible set of strategies S. Similarly, player 2 selects a strategy $y \in T$. According to the choice of strategies an amount $K(x, y)$, the so-called payoff, is transferred from player 2 to player 1. In the primal problem (1), player 1 selects a (cautious) strategy for which this player is sure to receive at least the amount z_0. Player 2 selects in the dual problem (2) a strategy such that w_0 is a guaranteed maximum amount to be paid to player 1.

Interesting cases are obtained when optimal solutions exist for both problems such that $z_0 = w_0$. In this case, we speak of *strong duality*. In general, we have the following so-called weak duality.

Proposition 1: $z_0 \le w_0$.

Proof: For any $x_1 \in S$ we have $K(x_1, y) \le \max_{x\in S} K(x, y)$. Hence

$$\min_{y \in T} K(x_1, y) \le \min_{y \in T} \max_{x \in S} K(x, y).$$

Since $x_1 \in S$ is arbitrarily selected we get

$$\max_{x \in S} \min_{y \in T} K(x, y) \le \min_{y \in T} \max_{x \in S} K(x, y),$$

i.e. $z_0 \le w_0$.

When strong duality does not exist, i.e. when $z_0 < w_0$ we speak of a *duality gap* between the dual problems.

Closely related to strong duality is the following notion.

$(x_0, y_0) \in S \times T$ is a *saddle point* provided that

$$K(x, y_0) \le K(x_0, y_0) \le K(x_0, y) \text{ for all } (x, y) \in S \times T.$$

The next proposition states the relationship.

Proposition 2: $(x_0, y_0) \in S \times T$ *is a saddle point if and only if:*

(i) x_0 *is an optimal solution of the primal problem,*
(ii) y_0 *is an optimal solution of the dual problem and*
(iii) $z_0 = w_0$.

Proof: By a reformulation of the definition, we get that (x_0, y_0) is a saddle point if and only if

$$\min_{y \in T} K(x_0, y) = K(x_0, y_0) = \max_{x \in S} K(x, y_0).$$

This implies that

$$z_0 = \max_{x \in S} \min_{y \in T} K(x, y) \ge K(x_0, y_0) \ge \min_{y \in T} \max_{x \in S} K(x, y) = w_0.$$

In general, by the weak duality in proposition 1 $z_0 \le w_0$. Hence (x_0, y_0) is a saddle point if and only if

$$\max_{x \in S} \min_{y \in T} K(x, y) = K(x_0, y_0) = \min_{y \in T} \max_{x \in S} K(x, y).$$

This is equivalent to (i), (ii), and (iii) in the proposition.

In the above framework of a game, a saddle point constitutes an equilibrium point among all strategies.

In the sequel, we shall consider some specific types of optimization problems derived from the primal problem (1) and the dual problem (2) by further specifications of the sets S and T and the function $K(x, y)$.

Let $x \in \mathbb{R}^n$ be a variable, $f(x) : \mathbb{R}^n \to \mathbb{R}$ a function, $g(x) : \mathbb{R}^n \to \mathbb{R}^m$ a multivalued function and $b \in \mathbb{R}^m$ a vector of constants. Let $K(x, y) = f(x) + y(b - g(x))$ with $y \in \mathbb{R}^m_+$, i.e. T is defined as the non-negative orthant of $\mathbb{R}^m$. The primal problem then has the form

$$\max_{x\in S}\min_{y\geq 0} f(x) - yg(x) + yb. \tag{3}$$

Observe that if for a given x an inequality of $g(x) \leq b$ is violated, then the inner minimization in the primal problem (3) is unbounded. Hence, the corresponding x can be neglected as a candidate in the outer maximization. Otherwise, if a given $x \in S$ satisfies $g(x) \leq b$ we say that x is a *feasible solution*. In this case, the inner minimization over $y \in \mathbb{R}^m_+$ yields $f(x)$.

Hence, the primal problem may be converted to

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t} \quad & g(x) \leq b \\ & x \in S. \end{aligned} \tag{4}$$

The notation “s.t.” stands for “subject to”. The above is a standard format for a general *mathematical programming problem* consisting of an *objective function* to be maximized such that the optimal solution found is feasible. In the sequel we shall study various types of mathematical programming problems, first by a discussion of the classical case of convex programming. For each type we shall create the specific dual problem to be derived from the general dual (2).

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Biographical Sketch

Jørgen Tind is Professor of Operations Research in the Department of Statistics and Operations Research, Institute for Mathematical Sciences, University of Copenhagen, Denmark. He received his master's degree at University of Copenhagen and his senior doctor's degree at the University of Aarhus, Denmark. He was previously Associate Professor at University of Aarhus, and has held visiting positions at Cornell University, Université Catholique de Louvain, Carnegie-Mellon University and University of Linköping. He has done research in theory, modeling and solution methods in the area of optimization. He has particularly been involved in topics at the interface between mathematics and economics. He is engaged in teaching programs, especially devoted to subjects regarding this interface. His works have been published in international scientific journals dealing with theory and applications in optimization, such as *Mathematical Programming, Management Science, Operations Research Letters, Journal of Global Optimization* and others. He has organized several scientific meetings, and is Program Chairman for the 18th International Symposium on Mathematical Programming, Copenhagen, 2003.