NONCONVEX VARIATIONAL PROBLEMS

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Keywords: calculus of variations, compatibility, convex, direct method, envelope, minimizer, minimizing sequence, oscillations, polyconvex, probability, nonconvex, quasiconvex, rank one convex, relaxation, Young measure, vector valued, wells.

Contents

Summary

This note is concerned with nonconvex problems of the calculus of variations. First, the notion of minimizer is introduced through simple examples. The direct method of finding a minimizer and its limitations in the nonconvex case are then explained. We present also shortly the so-called relaxation theory in the scalar and vectorial case. Finally, we develop some techniques useful to gain information for problems with no minimizer.

1. Introduction

A lot of problems in physics can be formulated as minimization problems, i.e. one minimizes some quantity, for instance some energy, on a suitable class of functions and one looks for a point achieving the infimum. This is the case for instance in elasticity theory and problems in this field where the deformation of the body is searched as a minimizer of a certain functional. Let us recall what is meant by a minimizer.

Definition 1

Let C be a set and φ a function from C into $\mathbb{R}$. $m \in C$ is a minimizer of φ on C if it holds

$$\inf_{x \in C} \varphi(x) = \varphi(m). \qquad (1)$$

Note that the existence of a minimizer imposes to φ to be bounded from below.

For a function φ and an arbitrary set various situations can occur. The archetype of the situation is already clear for $C = \mathbb{R}$, i.e. for a function from $\mathbb{R}$ into $\mathbb{R}$. Indeed, we can have:

- φ has a unique minimizer on $\mathbb{R}$. This is the case for instance for the function $\varphi(x) = x^2$.
- φ has infinitely many minimizers. This is the case for $\varphi(x) = (x - 1)^+$. $(\ )^+$ denotes the positive part of functions.
- φ has no minimizer. This is the case for instance for $\varphi(x) = e^{-x}$. The infimum of this function on $\mathbb{R}$ is 0, but there is no point where it is achieved.

When one minimizes on classes of functions the same situations arise, although sometimes it is not so easy to see it. However, there is no reason for the situation occurring in dimension 1 not to be reconducted in higher dimensions. Let us show this through a simple example.

Let Ω be the interval (0, 1). Let

$$C = \left\{ v \in W^{1,\infty}(0,1) \,\middle|\, v(0) = 0, v(1) = 1/2 \right\} \qquad (2)$$

where $W^{1,\infty}(0, 1)$ denotes the set of Lipschitz continuous functions on Ω. Then we can show

Theorem 1.1

The problems

$$\inf_{v \in C} \int_\Omega v'^2(x) dx \qquad (3)$$

$$\inf_{v \in C} \int_\Omega (1 - v'^2(x))^2 dx \qquad (4)$$

$$\inf_{v \in C} \int_\Omega \operatorname{Arc}\tan^2 v'(x) dx \qquad (5)$$

have respectively, a unique minimizer, infinitely many minimizers, and no minimizer.

Proof.

Let us start with (3). Then we claim that

$$u(x) = \frac{1}{2}x \tag{6}$$

is the only minimizer. To see it, it is enough to remark that for any $u \in C$

$$\begin{aligned} \int_0^1 v'^2\, dx - \int_0^1 v'^2\, dx &= \int_0^1 (v'-u')\cdot(v'+u')dx \\ &= \int_0^1 (v'-u')^2\, dx + \int_0^1 v'-u'\; dx \\ &= \int_0^1 (v'-u')^2\, dx \geq 0. \end{aligned} \tag{7}$$

Now this last integral vanishes if and only if $(v-u)' \equiv 0 \Rightarrow v-u = \text{cst}$, but due to our boundary conditions this is if and only if $v \equiv u$.

For (4), it is clear the infimum 0 is achieved for any function $u \in C$ such that $u' = \pm 1$. They are infinitely many such functions. Each of them uses one path made of broken lines of slope ±1, and located in the rectangle with vertices (0, 0), (3/4, 3/4), (1, 1/2), (1/4, –1/4).

For (5), if we show that the infimum is 0 then we are done. Indeed, if u is a minimizer, we must have $u' = 0$ identically, hence $u = \text{cst}$, which is impossible due to the boundary conditions to be matched. Now, to show that the infimum is 0 we consider the "sequence" of functions u_ε defined by

$$u_\in = \begin{cases} 0 \text{ on } (0, 1-\varepsilon), \\ \dfrac{1}{2\varepsilon}(x-1+\varepsilon) \text{ on } (1-\varepsilon, 1). \end{cases} \tag{8}$$

Clearly, since $Arctan^2(v)$ is bounded from above by $\pi^2/4$ one has

$$0 \leq \int_\Omega Arctan^2 u_\varepsilon'(x)\, dx \leq \varepsilon \frac{\pi^2}{4} \to 0 \text{ with } \varepsilon \to 0. \tag{9}$$

This completes the proof of the theorem.

Remark 2

The functional (3) *is convex, the functionals* (4), (5) *are not.*

2. The Direct Method of the Calculus of Variations

Suppose that we consider a function

$$\varphi : X \to \mathbb{R} \tag{10}$$

where X is some metric space. If C is a subset of X then we would like to find u such that

$$u \in C, \text{ and } \varphi(u) = \inf_C \varphi(v). \tag{11}$$

The direct method of the Calculus of Variations is the most natural method that one can think of in order to solve (11) that is to say one considers a minimizing sequence in C–i.e. a sequence u_n such that

$$u_n \in C, \qquad \varphi(u_n) \to \inf_{v\in C} \varphi(v) \tag{12}$$

and one tries to show that u_n converges toward u the solution to (11). The two difficulties are to show that

- u_n converges toward $u \in C$ (13)
- u is a minimizer of the problem . (14)

A simple and frequent situation in the Calculus of Variations where this can be achieved is the following:

Theorem 3

Let X be a reflexive Banach space. Let C be a weakly closed subset of X and $\varphi : C \to \mathbb{R}$ *be a function such that*

- φ has a bounded minimizing sequence in C (15)
- φ is weakly lower semicontinuous $(l.s.c.)$ on C (16)

then there exists u such that

$$\varphi(u) = \inf_{v\in C} \varphi(v). \tag{17}$$

Proof

Let us denote by u_n a minimizing sequence (see (12)) which is bounded. We can extract a subsequence that we still label by u_n such that u_n converges toward u weakly. C being weakly closed and φ weakly $l.s.c$ we have

$$u \in C, \ \inf_{v\in C} \varphi(v) = \lim \inf_{n\to+\infty} \varphi(u_n) \ge \varphi(u) \tag{18}$$

and the result follows.

Remark 4

The assumption (15) *holds for instance in the cases*

- *C is bounded* (19)

- $\lim_{\|x\|\to+\infty,\, x\in C} \varphi(x) = +\infty$ (20)

(*this last condition is often called coerciveness*).

The assumption (14) holds when φ is convex and continuous. Finally, C convex closed implies that C is weakly closed.

Example 5

For $f \in L^2(\Omega)$, Ω *a bounded open subset of* $\mathbb{R}^n$ *, there exists a unique u minimizing*

$$\varphi(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} fv dx \tag{21}$$

on $H_0^1(\Omega)$. *u is the so-called solution to the Dirichlet problem. More generally, under some convexity assumptions on* ψ *we can consider the problem of minimizing*

$$\varphi(v) = \int_{\Omega} \psi(x, v(x), \nabla v(x)) dx \tag{22}$$

on some closed convex set of $H^1(\Omega)$.

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Biographical Sketch

Michel Chipot is currently a professor at the University of Zurich. He graduated from the University of Paris 6 (these d'etat 1981) under the supervision of Professor H. Brezis. Before Zurich he occupied various positions in France, first as assistant professor in Nancy and then in Metz as professor from 1985 to 1995. Meanwhile he occupied several visiting positions in the United States (Brown University, 1981-82, the University of Maryland, 1983-1984, the University of Minnesota, 1984-1985, 1987, 1990, and Carnegie Mellon University, 1987). He is the author of two books and of more than hundred publications in the fields of PDEs, Calculus of Variations, and Numerical Analysis.