SYMMETRY PRINCIPLES AND CONSERVATION LAWS

J.L. Díaz-Cruz and **A. Rosado**
Benemérita Universidad Autónoma de Puebla, México

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Contents

Summary

Symmetry considerations provide an extremely powerful and useful tool in our efforts to discover and understand the order in nature. Gradually they have become the backbone of our theoretical formulation of the physical laws. One of the most important consequences of the existence of symmetry (or invariance) is that a corresponding conservation law, with a conserved charge, follows. We shall elaborate this statement and its consequences, mainly within the context of contemporary physics.

1. Introduction

Since the beginning of physics, symmetry considerations have provided us an extremely powerful and useful tool in our effort to understand nature. Gradually they have become the backbone of our theoretical formulation of physical laws. It is well known that the principles of symmetry play an important role in physics. Furthermore, if a symmetry (or an invariance) exists, then the conservation of a corresponding quantity follows. For instance, from the fact that space-time is invariant to parallel motion, the conservation laws of energy and momentum are created, and from the symmetry of direction or rotations, the conservation of angular momentum is born. Symmetries under reflection in space and under exchange of particle and antiparticle correspond to conservation of *parity* (P) and *charge-conjugation* (C), respectively. The approximate symmetry of the u and d quarks leads to the approximate conservation of isospin, and so forth.

Thinking about what symmetry means in such cases as above we realize that it refers to a situation in which there exist several different states that, however, are basically equivalent to each other with respect to the laws of physics; symmetries imply that the laws of physics are invariant to the symmetry operations in which a state is replaced by an equivalent state.

In such cases, there is a need to distinguish between two equivalent states. In Quantum Mechanics such states are denoted as : $|A\rangle$ and $|A'\rangle$ (using the Dirac notation). Then, it suffices to appropriately express the operation which changes $|A\rangle$ to $|A'\rangle$, and the quantities used in such expressions are associated with the conserved quantities, or quantum numbers.

If physical laws possess a symmetry, existence of an arbitrary state $|A\rangle$ would imply that any equivalent state formed by a symmetry operation must also be possible. For example, an electron with spin component $S_z = +\hbar/2$ can be placed in a magnetic field along $\hat{z}$, as well as in a rotated field along $\hat{z}'$. Both states are, in principle, possible, and the characteristics of the electron do not depend on the direction.

There are four main groups of symmetries that are found to be of importance in physics:

1. Continuous space-time symmetries, such as translation, rotation, acceleration, etc.
2. Permutation symmetry: Bose-Einstein and Fermi-Dirac statistics.
3. Discrete symmetries, such as space inversion, time reversal, particle-antiparticle conjugation, etc.
4. Unitary (internal) symmetries, which include:

   (a) *U*(1)-symmetries, such as those related to the conservation of electric charge, baryon number, lepton number, etc.,
   (b) *SU*(2) (isospin)-symmetry,
   (c) *SU*(3) (color)-symmetry,
   (d) *SU*(n) (flavor)-symmetry.

Among these, the first two groups, together with some of the *U*(1)-symmetries and perhaps the *SU*(3) (color)-symmetry in the last group, are believed to be exact. All the rest seems to be broken.

2. Symmetries and Conservation Laws; Noether's Theorem

The root of all symmetry principles lies in the assumption that it is impossible to observe certain basic quantities; these will be called "non-observable". Let us illustrate the relation between non-observables, symmetry transformations and conservation laws by simple examples. Consider the interaction energy *V* between two particles at position $\vec{r}_1$ and $\vec{r}_2$. The assumption that the *absolute position is a non-observable* means that we can arbitrarily choose the origin *O* from which these position vectors are drawn; the interaction energy should be independent of O. In other words, V is invariant under an arbitrary space translation, changing O to O';

$$\vec{r}_1 \rightarrow \vec{r}_1 + \vec{d} \text{ and } \vec{r}_2 \rightarrow \vec{r}_2 + \vec{d} \tag{1}$$

consequently, V is a function only of the relative distance $|\vec{r}_1 - \vec{r}_2|$:

$$V = V\left(|\vec{r}_1 - \vec{r}_2|\right). \tag{2}$$

From this, we deduce that the total momentum of this system of two particles must be conserved, since its rate of change is equal to the force

$$-\left(\vec{\nabla}_1 + \vec{\nabla}_2\right)V$$

which, on account of (2), is zero.

This example illustrates the interdependence among three aspects of a symmetry principle: the physical existence of a non-observable, the implied invariance under the connected mathematical transformation and the physical consequence of a conservation law, which can be stated as a selection rule.

In an entirely similar way, we may assume the absolute time to be a non-observable. The physical law must then be invariant under a time translation

$$t \rightarrow t + \tau$$

which results in the conservation of energy. By assuming the absolute spatial direction to be a non-observable, we derive rotation invariance and obtain the conservation law of angular momentum. By assuming that absolute (uniform) velocity is not an observable, we derive the symmetry requirement of Lorentz invariance, and with it the conservation laws connected with the six generators of the Lorentz group. Similarly, the foundation of general relativity rests on the assumption that it is impossible to distinguish the difference between acceleration and a suitably arranged gravitation field.

Thus, invariance under translations, time displacements, rotations, and Lorentz transformations leads to the conservation of momentum, energy, and angular momentum. Besides these conservation laws, associated with "external" symmetries, there are also "internal" symmetry transformations, that do not mix fields with different space-time properties, that is transformations that commute with the space-time components of the wavefunction.

For example, an electron is described by an array of complex fields (ψ) and the corresponding Lagrangian can be written as follows

$$\mathcal{L} = \frac{i}{2}\left[\bar{\psi}\gamma_\mu\, \partial^\mu \psi - \left(\partial^\mu \bar{\psi}\right)\gamma_\mu \psi\right] - m\bar{\psi}\psi\,. \tag{3}$$

The Lagrangian (3) is invariant under the phase transformation

$$\psi \to e^{i\alpha}\psi , \tag{4}$$

where α is a real constant. This can be easily checked by noting that $\partial_\mu \psi \to e^{i\alpha}\partial_\mu \psi$ and $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$.

The family of phase transformation $U(\alpha) \equiv e^{i\alpha}$, where a single parameter α may run continuously over real numbers, forms a unitary Abelian group known as the $U(1)$ group. Abelian just records the property that the group multiplication is commutative:

$$U(\alpha_1)U(\alpha_2) = U(\alpha_2)U(\alpha_1). \tag{5}$$

One may think that the existence of $U(1)$ invariance of $\mathcal{L}$ is rather trivial and unimportant. This is not so. Through Noether's theorem, it implies the existence of a conserved current. To see this, it is sufficient to study the invariance of $\mathcal{L}$ under an infinitesimal $U(1)$ transformation,

$$\psi \to (1+i\alpha)\psi . \tag{6}$$

Invariance requires the Lagrangian to be unchanged, that is,

$$\begin{aligned}
0 &= \delta\mathcal{L} \\
&= \frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta(\partial_\mu \psi) + \delta\bar{\psi}\frac{\partial \mathcal{L}}{\partial \bar{\psi}} + \delta(\partial_\mu \bar{\psi})\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \\
&= \frac{\partial \mathcal{L}}{\partial \psi}(i\alpha\psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(i\alpha\partial_\mu \psi) + \dots \\
&= i\alpha\left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right]\psi + i\alpha\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi\right) + \dots
\end{aligned} \tag{7}$$

The term in square brackets vanishes by virtue of the Euler-Lagrange equation for ψ, *i.e.*

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{8}$$

(and similarly for $\bar{\psi}$), and so (7) reduces to the form of an equation for a conserved current:

$$\partial_\mu j^\mu = 0 \tag{9}$$

where

$$j^{\mu} = \frac{ie}{2}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)}\psi - \bar{\psi}\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\bar{\psi}\right)} \right) = -e\bar{\psi}\gamma^{\mu}\psi\,, \qquad (10)$$

using (3). The proportionality factor is chosen so the j^{μ} matches up with the electromagnetic charge current density of an electron of charge *–e*. It follows from (9) that the total charge

$$Q = \int d^3\ x\ j^0 \qquad (11)$$

must be a conserved quantity because of the $U(1)$ phase invariance. Thus, we have arrived at the mathematical expression that relates the existence of conserved charges (Q) and the existence of symmetries— a result that has a profound impact in physics. In Table 1, we summarize these three fundamental aspects for some of the symmetry principles used in physics.

Non-observables	Symmetry Transformations	Conservation of Laws or Selection Rules
difference between identical particles	permutation	Bose–Einstein or Fermi–Dirac statistics
absolute spatial position	space translation $\vec{r} \to \vec{r} + \vec{d}$	momentum
absolute time	time translation $t \to t + \tau$	energy
absolute spatial direction	rotation $\hat{r} \to \hat{r}'$	angular momentum
absolute right (or absolute left)	$\vec{r} \to -\vec{r}$	parity
absolute sign of electric charge	$e \to -e\left(\text{or } \psi \to e^{i\alpha}\psi^{\dagger}\right)$	charge conjugation
relative phase between states of different charge *Q*	$\psi \to e^{iQ\theta}\psi$	charge
relative phase between states of different baryon number *N*	$\psi \to e^{iN\theta}\psi$	baryon number
relative phase between states of different lepton number *L*	$\psi \to e^{iL\theta}\psi$	lepton number
difference between different coherent mixture of *p* and *n* states	$\left(\frac{p}{n}\right) \to u\left(\frac{p}{n}\right)$	isospin

Table 1: Summary of fundamental aspects for some of the symmetry principles used in physics

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Biographical Sketches

J. Lorenzo Díaz-Cruz was born in 1961 in Guerrero, México. He got his Ph.D. at the University of Michigan (1989), afterwards he spent as a posdoct two years at the Universidad Autónoma de Barcelona (1990-1992). Since 1994 he has been full professor at Universidad Autónoma de Puebla (Institute of Physics and Faculty of Physics and Mathematics). His line of research is on Supersymmetry (Theory, models and phenomenology).

Alfonso Rosado-Sánchez was born in 1958 in Matías Romero, Oaxaca, México. He got his Ph.D. at the Physics Department of Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional (CINVESTAV-IPN) in México City (1984), afterwards he spent as a posdoct two years at the Physikalisches Institut der Universität Würzburg in Germany (1985-1987). During 1987-1991 was professor at the Physics Department of CINVESTAV-IPN. Since 1991 has been full professor at Institute of Physics of Universidad Autónoma de Puebla. His line of research is on phenomenology of the strong and electroweak interactions.