LINEAR ELASTO-STATICS

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Summary

Governing equations are developed for the displacements and stresses in a solid with a linear constitutive law under the restriction that strains are small. Alternative variational formulations are introduced which can be used to obtain approximate analytical solutions and which are also used to establish a uniqueness theorem. Techniques for solving boundary-value problems are discussed, including the Airy stress function and Muskhelishvili's complex-variable formulation in two dimensions and the Papkovich-Neuber solution in three dimensions. Particular attention is paid to singular stress fields due to concentrated forces and dislocations and to geometric discontinuities such as crack and notch tips. Techniques for solving two-dimensional problems for generally anisotropic materials are briefly discussed.

1. Introduction

The subject of Elasticity is concerned with the determination of the stresses and displacements in a body as a result of applied mechanical or thermal loads, for those cases in which the body reverts to its original state on the removal of the loads. If the loads are applied sufficiently slowly, the particle accelerations will be small and the body will pass through a sequence of equilibrium states. The deformation is then said to be 'quasi-static'. In this chapter, we shall further restrict attention to the case in which the stresses and displacements are linearly proportional to the applied loads and the strains and rotations are small. These restrictions ensure that the principle of linear superposition applies — i.e.,, if several loads are applied simultaneously, the resulting stresses and displacements will be the sum of those obtained when the loads are applied separately to the same body. This enables us to employ a wide range of series and transform techniques which are not available for non-linear problems.

1.1. Notation for Position, Displacement and Strain

We shall define the position of a point in three-dimensional space by the Cartesian coordinates (x_1, x_2, x_3). Latin indices i, j, k, l, etc will be taken to refer to any one of the values 1,2,3, so that the symbol x_i can refer to any one of x_1, x_2, x_3. The Einstein summation convention is adopted for repeated indices, so that, for example

$$x_i x_i \equiv \sum_{i=1}^{3} x_i x_i = x_1^2 + x_2^2 + x_3^2 = R^2 \, , \qquad (1)$$

where R is the distance of the point (x_1, x_2, x_3) from the origin. We can also define position using the *position vector*

$$\boldsymbol{R} = \boldsymbol{e}_i x_i \, , \qquad (2)$$

where e_i is the unit vector in direction x_i.

Suppose that a given point is located at $\boldsymbol{R} = \boldsymbol{e}_i x_i$ in the undeformed state and moves to the point $\boldsymbol{\rho} = \boldsymbol{e}_i \xi_i$ after deformation. We can then define the *displacement vector* $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{\rho} - \boldsymbol{R} \, , \qquad (3)$$

or in terms of components,

$$u_i = \xi_i - x_i \, . \qquad (4)$$

When the body is deformed, different points will generally experience different displacements, so $\boldsymbol{u}$ is a function of position. We shall always refer displacements to the undeformed position, so that u_i is a function of x_1, x_2, x_3.

1.2. Rigid-Body Displacement

There exists a class of displacements that can occur even if the body is rigid and hence incapable of deformation. An obvious case is a rigid body translation $u_i = C_i$, where C_i are constants (independent of position). We can also permit a small rotation about each of the three axes (recall that in the linear theory rotations are required to be small). The most general rigid-body displacement field can then be written as:

$$u_i = C_i + D_j \epsilon_{ijk} x_k \ , \tag{5}$$

where ϵ_{ijk} is the *alternating tensor* which is defined to be 1 if the indices are in cyclic order (e.g. 1,2,3 or 2,3,1), –1 if they are in *reverse* cyclic order (e.g. 2,1,3) and zero if any two indices are the same.

1.3. Strain, Rotation and Dilatation

In the linear theory, the strain components e_{ij} can be defined in terms of displacements as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) . \tag{6}$$

This leads (for example) to the definitions

$$e_{11} = \frac{\partial u_1}{\partial x_1} ; \quad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \tag{7}$$

for normal and shear strains respectively. We also define the *rotation*

$$\omega_k = \frac{1}{2}\left(\epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \right) . \tag{8}$$

It can be verified by substitution that if the displacement is given by (5), there is no strain ($e_{ij} = 0$) and the rotation $\omega_k = D_k$.

By considering the deformation of an infinitesimal cube of material of initial volume V , it can be shown that the proportional change in volume is the sum of the three normal strains — i.e.,

$$\frac{\delta V}{V} \equiv e = e_{ii} \ . \tag{9}$$

This quantity is known as the *dilatation* and is denoted by the symbol e .

1.4. Compatibility of Strain

If the strains e_{ij} and rotations ω_k are known functions of x_1, x_2, x_3, Eqs. (6, 8) constitute a set of partial differential equations that can be integrated to obtain the displacements u_i . In a formal sense, one can write

$$\boldsymbol{u}_B = \boldsymbol{u}_A + \int_A^B \frac{\partial \boldsymbol{u}}{\partial S} dS \ , \qquad (10)$$

where A, B are two points in the body and the integration is performed along any line between A and B that is entirely contained within the body. The displacement u must be a single-valued function of position and hence the integral in Eq. (10) must be path-independent. Using Eqs. (6, 8) to define the derivatives inside this integral, it can be shown that this requires that the strains satisfy the *compatibility equations*

$$\epsilon_{pks} \frac{\partial}{\partial x_k} \left(\frac{\partial e_{sj}}{\partial x_i} - \frac{\partial e_{si}}{\partial x_j} \right) = 0 \ . \qquad (11)$$

Alternatively, these equations can be obtained by eliminating the displacement components between Eqs. (6). Equation (11) can be expanded to give three equations of the form

$$\frac{\partial^2 e_{11}}{\partial x_2{}^2} + \frac{\partial^2 e_{22}}{\partial x_1{}^2} = 2 \frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} \qquad (12)$$

and three of the form

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3} \left(\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} - \frac{\partial e_{12}}{\partial x_3} \right). \qquad (13)$$

The compatibility equations are sufficient to ensure that the integral in (10) is single-valued if the body is simply connected, but if it is multiply connected, they must be supplemented by the explicit statement that the corresponding integral around a closed path surrounding any hole in the body be zero. Explicit forms of these additional conditions in terms of the strain components were developed by E.Cesaro and are known as *Cesaro integrals*.

2. Traction and Stress

We shall use the term *traction* and the symbol $\boldsymbol{t}$ to define the limiting value of force per unit area applied to a prescribed infinitesimal elementary area, such as a region of the boundary of the body. Since the loaded surface is defined, the traction is a vector t_i. To define a component of *stress* $\boldsymbol{\sigma}$ within the body, we need to identify both the plane on which the stress component acts and the direction of the traction on that plane. We define the plane by its outward normal, sothat the x_i-plane is perpendicular to the direction x_i. Notice that this plane can also be defined as the locus of all points (x_1, x_2, x_3) satisfying the equation $x_i = C$ where C is any constant. With this notation, we then define the stress component σ_{ij} as the component of traction in the j-direction acting on the x_i-plane. The resulting components are illustrated in Figure 1.

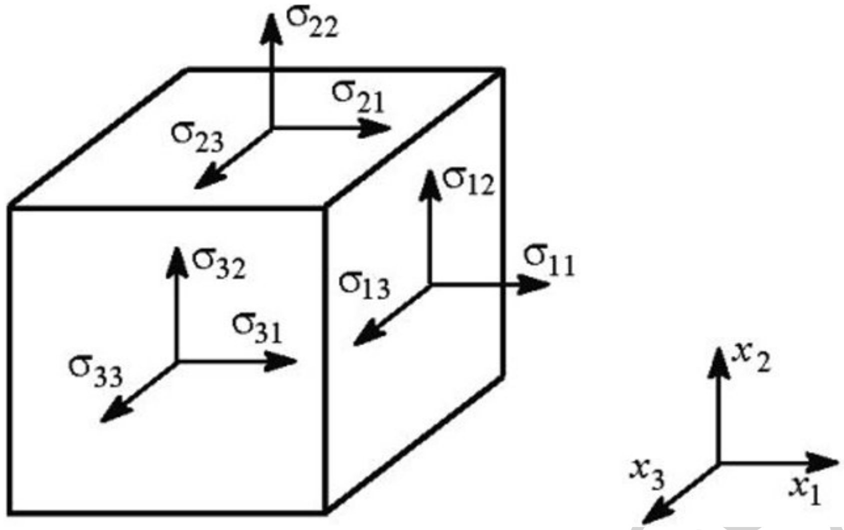

Figure 1. Notation for stress components.

The equilibrium of moments acting on the block in Figure 1.1 requires that $\sigma_{ij} = \sigma_{ji}$ and hence that the matrix of stress components

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \quad (14)$$

is symmetric. Notice that the diagonal elements of the stress matrix define *normal stresses* and the convention implies that tensile normal stresses are positive. The off-diagonal elements define *shear stresses*.

2.1. Equilibrium of Stresses

The stress components in any continuum are constrained by the requirement that all parts of the body obey Newton's law of motion. Applying this condition to an infinitesimally small rectangular element of material $(\delta x_1 \delta x_2 \delta x_3)$, we obtain

$$\frac{\partial \sigma_{ij}}{\partial x_j} + p_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (15)$$

where p_i represents the components of a body force vector $\boldsymbol{p}$ per unit volume, ρ is the density and t is time. The basic postulate of elasto-statics is that the loading rate is sufficiently small for the acceleration term $\partial^2 u_i / \partial t^2$ to be neglected, leading to the equilibrium equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + p_i = 0\,. \quad (16)$$

3. Transformation of Coordinates

If x_1, x_2, x_3 and x_1', x_2', x_3' are two sets of Cartesian coordinates sharing the same origin, we can define a matrix $\boldsymbol{l}$ of *direction cosines* such that l_{ij} is the cosine of the angle between the axes x_i' and x_j. It then follows that

$$x_i' = l_{ij} x_j \;;\quad x_i = l_{ji} x_j' \tag{17}$$

and since the three rows and three columns of l each defines a set of orthogonal unit vectors, we also have

$$l_{ij} l_{ik} = \delta_{jk} \;;\quad l_{ij} l_{kj} = \delta_{ik} \;. \tag{18}$$

Vectors, such as the displacement u can be transformed to and from the new coordinate system by the relation

$$u_i' = l_{ij} u_j \;;\quad u_i = l_{ji} u_j' \;. \tag{19}$$

and the strain components e_{ij} transform according to the rules

$$e_{ij}' = l_{ip} l_{jq} e_{pq} \;;\quad e_{ij} = l_{pi} l_{qj} e_{pq}' \;, \tag{20}$$

which follow from the definitions (6) and (17, 19). The corresponding stress transformation equations are obtained by considering the equilibrium of an infinitesimal tetrahedron whose four surfaces are perpendicular to x_1, x_2, x_3, x_i' respectively. We obtain

$$\sigma_{ij}' = l_{ip} l_{jq} \sigma_{pq} \;;\quad \sigma_{ij} = l_{pi} l_{qj} \sigma_{pq}' \;, \tag{21}$$

which of course have the same form as (20). Quantities which transform according to equations of this form are known as *Cartesian tensors* of rank 2.

4. Hooke's Law

Linear elasticity is restricted to materials that obey Hooke's law in the sense that the stress and strain tensors are linearly related. The most general such relation can be written as:

$$\sigma_{ij} = c_{ijkl} e_{kl} = c_{ijkl} \frac{\partial u_k}{\partial x_l} \;, \tag{22}$$

where c_{ijkl} is a Cartesian tensor of rank 4 known as the *elasticity tensor*. It can be transformed into the coordinate system x_1', x_2', x_3' using the relation

$$c'_{ijkl} = l_{ip} l_{jq} l_{kr} l_{ls} c_{pqrs} \ . \tag{23}$$

Equation (22) can be viewed as a set of linear algebraic equations for e_{kl}, which can be inverted to give an equation of the form

$$e_{ij} = s_{ijkl} \sigma_{kl} \ , \tag{24}$$

where s_{ijkl} is known as the *compliance tensor*.

Both the elasticity tensor and the compliance tensor must satisfy the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij} = c_{ijlk} \ , \tag{25}$$

which follow from (i) the symmetry of the stress and strain tensors (e.g. $\sigma_{ij} = \sigma_{ji}$) and (ii) the reciprocal theorem, which we shall discuss in Section 6.4 below. Using these conditions, the maximum number of independent constants in c_{ijkl} is reduced to 21. However, the material may have additional structural symmetries in particular coordinate systems which further reduces the number of independent elastic constants. The greatest degree of symmetry arises when the material is *isotropic* so that the elasticity tensor is invariant under all Cartesian coordinate transformations. In this case, only two constants are independent and they can be defined such that

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} \right), \tag{26}$$

where λ, μ are *Lamé's constants*. The elastic constitutive law (22) then takes the form

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} = \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{27}$$

An alternative statement of the isotropic constitutive law is

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E} \ , \tag{28}$$

where E, ν are *Young's modulus* and *Poisson's ratio* respectively. Clearly the two sets of elastic constants are related, since (28) is the inversion of (27). In fact

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)}; \quad \mu = \frac{E}{2(1+\nu)}; \quad E = \frac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)}; \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{29}$$

4.1. Equilibrium Equations in Terms Of Displacements

Hooke's law (22) and the strain displacement relations (6) can be used to write the equilibrium equations (16) in terms of the displacements, giving

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + p_i = 0 \,. \tag{30}$$

If the material is isotropic, we obtain

$$(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + p_i = 0 \,, \tag{31}$$

using (27) in place of (22).

5. Loading and Boundary Conditions

Suppose that an elastic body occupies the region Ω and that its boundary is denoted by Γ. In a typical problem, the body force p_i will be prescribed and the stress and displacement components are required to satisfy Eqs. (16, 30) respectively throughout Ω. In addition, three boundary conditions must be specified at each point on the boundary.

If the local outward normal to the surface is denoted by the unit vector $\boldsymbol{n}$ (i.e., n_i are the direction cosines of the normal), the corresponding traction t_i can be written as:

$$t_i = n_j \sigma_{ji} \,. \tag{32}$$

The boundary condition at any given point may comprise prescribed values of traction t_i or of displacement u_i or of some combination of the two. For example, if an elastic body is in contact with a plane frictionless rigid body defined by a normal in direction x_3, the normal displacement u_3 must be zero, and the two shear tractions

$$t_1 = \sigma_{31} \,; \quad t_2 = \sigma_{32} \tag{33}$$

must be zero. Notice that we cannot prescribe both the traction and the displacement in the same direction at any point, since this would generally lead to an ill-posed problem.

5.1. Saint-Venant's Principle

B.de Saint-Venant first enunciated the concept that if two systems of loading at a local region on a boundary are statically equivalent (i.e., they correspond to the same total force and moment) then their elastic stress fields will approach each other with increasing distance from the loaded region. An equivalent statement, appealing to the concept of superposition, is that a localized region of tractions that are self-equilibrated (i.e., they correspond to *zero* total force and moment) will cause a stress field that decays with increasing distance from the loaded region. This statement is generally known as Saint-Venant's principle. It cannot be proved and in fact there are some important exceptions, notably for the loading of thin-walled structures. For example, if a thin-walled cylindrical shell is pinched by a pair of equal and opposite forces at one end, the effects will penetrate a considerable distance along the axis of the shell. However, the principle can be extremely useful in other situations. For example, if two non-conforming elastic bodies are pressed together, a relatively complex stress field may be developed near the contact region, but at distances that are large compared with the contact area, the fields are well approximated by the solution for a concentrated force.

5.1.1. Weak Boundary Conditions

Saint-Venant's principle also permits us on occasion to obtain approximate solutions by replacing the true boundary conditions on a part of the boundary by *weak* boundary conditions, which state merely that the tractions in this region should have the same force and moment resultants as those in the actual problem. For example, suppose we seek to determine the stresses in the two-dimensional rectangular body $-a < x_1 < a, -b < x_2 < b$ and that the boundary conditions on $x_1 = a$ are

$$\sigma_{11}(a,x_2) = f_1(x_2)\,; \quad \sigma_{12}(a,x_2) = f_2(x_2)\,, \tag{34}$$

where f_1, f_2 are prescribed functions of x_2 in $-b < x_2 < b$. The weak boundary conditions equivalent to (34) are

$$F_1 \equiv \int_{-b}^{b} \left(\sigma_{11}(a,x_2) - f_1(x_2)\right) dx_2 = 0\,; \quad F_2 \equiv \int_{-b}^{b} \left(\sigma_{12}(a,x_2) - f_2(x_2)\right) dx_2 = 0$$

$$M \equiv \int_{-b}^{b} \left(\sigma_{11}(a,x_2) - f_1(x_2)\right) x_2 dx_2 = 0\,, \tag{35}$$

where F_1, F_2, M are the force and moment resultants on the boundary implied by the difference between a candidate stress field and one that exactly satisfies (34). Saint-Venant's principle implies that any solution satisfying (35) will differ significantly from the solution satisfying the *strong* (point wise) conditions (34) only in a region near $x_1 = a$ of magnitude comparable to the dimension b and hence if $a \gg b$, the solution will be quite accurate in a region distant from this boundary. We shall see in Section 8.1 below that this device often enables us to obtain closed-form approximations for problems that would otherwise be extremely complex.

5.2. Body Force

It is important to distinguish between loading of a body by surface tractions and by body force. A body force is an external force that applies in a distributed sense on the internal particles of the body. Thus, it must necessarily involve a physical mechanism that can 'act at a distance'. The commonest case of this kind involves gravitational forces (self weight), but other mechanisms are possible, such as electromagnetic forces.

Another important source of body force arises if the body experiences rotation or translational acceleration. It might be argued that this takes us beyond the field of elasto-statics, but a quasi-static solution can still be obtained if the acceleration terms included are only those corresponding to the rigid-body motion. For example, if the body is rotating at constant angular velocity Ω, D'Alembert's principle can be used to convert the corresponding centripetal acceleration into a centrifugal body force $\rho\Omega^2 r$, where r is the distance from the axis of rotation.

If the body forces are prescribed, they can be carried to the right hand side as known functions in Eqs. (16, 30), in which case these become inhomogeneous linear partial differential equations. The solution of these equations can then be constructed as the sum of any particular solution and the general solution of the corresponding homogeneous equation. However, the homogeneous equation is also the equation to be satisfied when there are no body forces. Thus, one strategy for solving problems with body forces is (i) to seek any particular solution of the equilibrium equation (without regard to the boundary conditions on Γ) and then 'correct' the boundary conditions by superposing an appropriately general solution of the problem without body force.

The particular solution is generally easy to obtain and can often be written down by inspection. For example, for the case of gravitational loading $p_i = -\rho g \delta_{i3}$, a simple particular solution of (31) is

$$u_1 = u_2 = 0 \, ; \quad u_3 = \frac{\rho g x_3^2}{2(\lambda + 2\mu)} \, . \tag{36}$$

For this reason, we shall mostly restrict the following discussion to problems without body force.

5.3. Thermal Expansion, Transformation Strains and Initial Stress

Elastic stresses can also be generated in a body as a result of internal physical processes that tend to change the parameters of the atomic or molecular structure. The simplest example is a change in temperature ΔT, which in the absence of stress would cause the body to expand equally in all three directions, giving the hydrostatic strain components

$$e_{ij} = \alpha \Delta T \delta_{ij} \, , \tag{37}$$

where α is the *coefficient of thermal expansion.* If the temperature is non-uniform, these strains may not satisfy the compatibility equation (11), in which case stresses will be induced so as to restore compatibility. Similar effects can be produced by other physical processes, such as a change in crystal structure as a material transforms from one phase to another. In the absence of stress, these processes (including thermal expansion) would contribute an 'inelastic' strain e_{ij}^0 which is additive to the elastic strain given by Hooke's law (24), giving

$$e_{ij} = s_{ijkl}\sigma_{kl} + e_{ij}^0 \, . \tag{38}$$

Practical objects are generally manufactured by some inelastic process. For example, a body may be solidified from an initially liquid state, or it may be plastically deformed into its final configuration. These processes typically leave the body in a state of *initial stress* or *residual stress*, meaning the state of stress that would remain in the body if all external loads were removed. Mathematically, there is no way to determine the residual stress without modeling the inelastic manufacturing process from which it derived. Various experimental techniques can be used to estimate the residual stresses in a body once manufactured. For example, X-ray diffraction can be used to estimate the mean atomic spacing and hence the elastic strain at various points on the surface of a body, from which the residual stresses can be deduced using Hooke's law.

If a body contains a non-zero residual stress field before loading, the stresses after loading will simply be the superposition of the residual stresses and the elastic stresses that would be induced in an initially stress-free body by the external loads. In the rest of this chapter, we shall therefore consider only the second of these two components. In other words, we shall assume that the unloaded body is free of stress.

6. Strain Energy and Variational Methods

When a body is deformed, the external forces do work. If the deformation is elastic, this work can be recovered on unloading and is therefore stored in the deformed body as *strain energy*. By considering the work done in gradually applying the stress components σ_{ij} to an infinitesimal rectangular element, we can show that the *strain energy density* — i.e., the strain energy stored per unit volume — is

$$U_0 = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} = \frac{1}{2}s_{ijkl}\sigma_{ij}\sigma_{kl} \tag{39}$$

and the strain energy stored in the entire body Ω is

$$U = \int_\Omega U_0 d\Omega\,. \tag{40}$$

Notice incidentally that U_0 must be positive for all possible states of stress or deformation and this places some inequality restrictions on the tensors c_{ijkl}, s_{ijkl}.

The same principle applies to an extended body with a non-uniform stress field. If the external loads are applied sufficiently slowly for accelerations (and hence kinetic energy) to be negligible, the work done during their application must be equal to the total strain energy in the body. This leads to the condition

$$\frac{1}{2}\int_\Omega p_i u_i d\Omega + \frac{1}{2}\int_\Gamma t_i u_i d\Gamma = \int_\Omega U_0 d\Omega\,. \tag{41}$$

We have argued here from the principle of conservation of energy, but this principle is implicit in Hooke's law, which guarantees that the work done on each infinitesimal particle by the body force and by the forces exerted by the surrounding particles is recoverable on unloading. Thus, Eq. (41) can be derived from the governing equations of elasticity without explicitly invoking conservation of energy. To demonstrate this, we first substitute (32) into the second term and apply the divergence theorem, obtaining

$$\frac{1}{2}\int_{\Gamma} t_i u_i d\Gamma = \frac{1}{2}\int_{\Gamma} n_j \sigma_{ji} u_i d\Gamma = \frac{1}{2}\int_{\Omega} \frac{\partial}{\partial x_j}(\sigma_{ji} u_i) d\Omega \,. \tag{42}$$

Differentiating by parts, we then have

$$\frac{1}{2}\int_{\Gamma} t_i u_i d\Gamma = \frac{1}{2}\int_{\Omega} \frac{\partial}{\partial x_j}(\sigma_{ji} u_i) d\Omega = \frac{1}{2}\int_{\Omega} \frac{\partial \sigma_{ji}}{\partial x_j} u_i d\Omega + \frac{1}{2}\int_{\Omega} \sigma_{ji} \frac{\partial u_i}{\partial x_j} d\Omega \,. \tag{43}$$

Finally, we use the equilibrium equation (16) in the first term and Hooke's law (22) in the second to obtain

$$\frac{1}{2}\int_{\Gamma} t_i u_i d\Gamma = -\frac{1}{2}\int_{\Omega} p_i u_i d\Omega + \frac{1}{2}\int_{\Omega} c_{ijkl} \frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_j} d\Omega \,, \tag{44}$$

from which (41) follows after using (39) in the last term.

6.1. Potential Energy of the External Forces

We can also construct a *potential energy* of the external forces which we denote by V . For a single concentrated force $\boldsymbol{F}$ moving through a displacement $\boldsymbol{u}$ this is defined as

$$V = -\boldsymbol{F}\cdot\boldsymbol{u} = -F_i u_i \;. \tag{45}$$

It follows by superposition that the potential energy of the boundary tractions and body forces is given by

$$V = -\int_{\Gamma_t} t_i u_i d\Gamma - \int_{\Omega} p_i u_i d\Omega \,, \tag{46}$$

where Γ_t is that part of the boundary over which the tractions are prescribed. We can then define the *total potential energy* Π as the sum of the stored strain energy and the potential energy of the external forces — i.e.,

$$\Pi = U + V = \frac{1}{2}\int_{\Omega} c_{ijkl} \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} d\Omega - \int_{\Gamma_t} t_i u_i d\Gamma - \int_{\Omega} p_i u_i d\Omega \,. \tag{47}$$

6.2. Theorem of Minimum Total Potential Energy

Suppose that the displacement field u_i satisfies the equilibrium equations (30) for a particular set of boundary conditions and that we then perturb this state by a small variation δu_i . The corresponding perturbation in Π is

$$\delta\Pi = \int_\Omega c_{ijkl} \frac{\partial \delta u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} d\Omega - \int_\Gamma t_i \delta u_i d\Gamma - \int_\Omega p_i \delta u_i d\Omega \,. \tag{48}$$

Notice that $\delta u_i = 0$ in any region Γ_u of Γ in which the displacement is prescribed and hence the domain of integration Γ_t in the second term on the right-hand side can be replaced by $\Gamma = \Gamma_u + \Gamma_t$.

Substituting for t_i from (32) and then applying the divergence theorem to the second term on the right-hand side of (48), we have

$$\begin{aligned} \int_\Gamma t_i \delta u_i d\Gamma &= \int_\Gamma \sigma_{ij} n_j \delta u_i d\Gamma = \int_\Omega \frac{\partial}{\partial x_j}\left(\sigma_{ij}\delta u_i\right) d\Omega \\ &= \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i d\Omega + \int_\Omega \frac{\partial \delta u_i}{\partial x_j} \sigma_{ij} d\Omega \,. \end{aligned} \tag{49}$$

Finally, using the equilibrium equation (16) in the first term and Hooke's law (22) in the second, we obtain

$$\int_\Gamma t_i \delta u_i d\Gamma = -\int_\Omega p_i \delta u_i d\Omega + \int_\Omega c_{ijkl} \frac{\partial \delta u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} d\Omega \,, \tag{50}$$

and comparing this with (48), we see that $\delta\Pi = 0$. In other words, the equilibrium equation requires that the total potential energy must be stationary with regard to any small variation δu_i in the displacement field u_i that is *kinematically admissible* — i.e., consistent with the displacement boundary conditions. A more detailed second order analysis shows that the total potential energy must in fact be a minimum and this is intuitively reasonable, since if some variation δu_i could be found which reduced Π, the surplus energy would take the form of kinetic energy and the system would not remain at rest.

6.2.1. Rayleigh-Ritz Approximations and the Finite Element Method

The theorem of minimum total potential energy provides a convenient strategy for the development of approximate solutions to problems where exact solutions are unavailable or overcomplicated. The first step is to define an approximation for the displacement field in the form

$$u_i(x_1, x_2, x_3) = \sum_{n=1}^{m} C_n f_i^{(n)}(x_1, x_2, x_3) \,, \tag{51}$$

where the $f_i^{(n)}$ are a set of approximating functions and C_n are arbitrary constants constituting the *degrees of freedom* in the approximation. The total potential energy is obtained from (47) as a quadratic function of the C_n and the theorem then requires that

$$\frac{\partial \Pi}{\partial C_n} = 0 \; ; \quad n = 1, m \, , \tag{52}$$

which defines m linear equations for the m unknown degrees of freedom C_n. The corresponding stress components can then be found by substituting (51) into Hooke's law (22).

If the approximating functions $f_i^{(n)}$ are defined over the entire body Ω, this typically leads to series solutions (e.g. power series or Fourier series) and the method is known as the *Rayleigh-Ritz method*. It is particularly useful in structural mechanics applications, but it is also useful for the challenging problem of the rectangular plate. However, if high accuracy is required it is often more effective to define a set of piecewise continuous functions each of which is zero except over some small region of the body. The body is divided into a set of *elements* and the displacement in each element is described by one or more *shape functions* multiplied by degrees of freedom C_n.

Typically, the shape functions are defined such that the C_n represent the displacements at specified points or *nodes* within the body. They must also satisfy the condition that the displacement be continuous between one element and the next for all C_n. Once the approximation is defined, Eq. (52) once again provides m linear equations for the m nodal displacements. This is the basis of the *finite element method*. Since the theorem of minimum total potential energy is itself derivable from Hooke's law and the equilibrium equation, an alternative derivation of the finite element method can be obtained by applying approximation theory directly to these equations. To develop a set of m linear equations for the C_n, we substitute the approximate form (51) into the equilibrium equations, multiply by m *weight functions*, integrate over the domain Ω and set the resulting m linear functions of the C_n to zero. The resulting equations will be identical to (52) if the weight functions are chosen to be identical to the shape functions.

6.3. Castigliano's Second Theorem

The strain energy U can be written as a function of the stress components, using the final expression in (39). We obtain

$$U = \frac{1}{2}\int_\Omega s_{ijkl}\sigma_{ij}\sigma_{kl} d\Omega \, . \tag{53}$$

If we now perturb the stress field by a small variation $\delta\sigma_{ij}$, the corresponding perturbation in U will be

$$\delta U = \int_\Omega s_{ijkl}\sigma_{kl}\delta\sigma_{ij} d\Omega = \int_\Omega \frac{\partial u_i}{\partial x_j}\delta\sigma_{ij} d\Omega \, . \tag{54}$$

The divergence theorem gives

$$\int_\Gamma u_i \delta\sigma_{ij} n_j d\Gamma = \int_\Omega \frac{\partial}{\partial x_j}\left(u_i \delta\sigma_{ij}\right) d\Omega = \int_\Omega \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij} d\Omega + \int_\Omega \frac{\partial \delta\sigma_{ij}}{\partial x_j} u_i d\Omega \tag{55}$$

and the second term on the right-hand side must be zero, since the stress perturbation $\delta\sigma_{ij}$ must satisfy the equilibrium equation (16) with no body force. Using (54, 55), we then have

$$\delta U = \int_{\Gamma_u} u_i \delta\sigma_{ij} n_j d\Gamma = \int_{\Gamma_u} u_i \delta t_i d\Gamma\,, \tag{56}$$

where the integral is taken only over Γ_u, since no perturbation in traction is permitted in Γ_t where t_i is prescribed. It follows that the *complementary energy*

$$C = U - \int_{\Gamma_u} u_i t_i d\Gamma \tag{57}$$

must be stationary with respect to all self-equilibrated variations of stress $\delta\sigma_{ij}$. This is *Castigliano's second theorem*. As with the Rayleigh-Ritz method, Castigliano's theorem can be used to obtain approximate solutions to otherwise intractable analytical problems. The first step is to define a self-equilibrated stress field containing an appropriate number of degrees of freedom C_i. This can often be done using an appropriate stress function, such as the Airy function of Section 7.2 or the Prandtl function of Section 7.4 below. The C_i are then determined by minimizing the complementary energy C of Eq. (57).

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Biographical Sketch

J.R.Barber graduated in Mechanical Sciences from Cambridge University in 1963 and joined British Rail, who later sponsored his research at Cambridge between 1965 and 1968 on the subject of thermal effects in braking systems. In 1969 he became Lecturer and later Reader in Solid Mechanics at the University of Newcastle upon Tyne. In 1981 he moved to the University of Michigan, where he is presently Arthur F. Thurnau Professor of Mechanical Engineering and Applied Mechanics and Professor of Civil and Environmental Engineering. He is a Chartered Engineer in the U.K., Fellow of the Institution of Mechanical Engineers, and has engaged extensively in consulting work in the field of stress analysis for engineering design. He is author of two books ('Elasticity' and 'Intermediate Mechanics of Materials') and over 160 articles in the fields of Elasticity, Thermoelasticity, Contact Mechanics, Tribology, Heat Conduction and Elasto-dynamics. He is a member of the editorial boards of the International Journal of Mechanical Sciences and the Journal of Thermal Stresses.