

## **HIERARCHY AND COMPLEXITY IN PHYSICAL SYSTEMS**

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### **Summary**

The discussion in this paper has to do with the uncertainty of knowledge, in particular regarding that paradigm of scientific certainty, physics. We argue that starting with the Greeks it has not been possible to predict events with absolute certainty, and even the contemporary laws of physics are in fact statistical and result in predictions that are probabilistic in character. We begin with the simple statistics of Gauss and the law of errors for independent measurements and demonstrate how the tying together of a hierarchy of scales in complex phenomena gives rise to more general statistical processes than envisioned by the early physical scientists. The dynamical behavior of complex processes is contained in the time series for physical observables. The coarse-graining of such time series data provides a renormalization group description of the dynamics of complex phenomena. These renormalization group relations describe how the hierarchy of scales in complex systems are interrelated. In particular, the Lévy distribution is shown to be a fixed point of a renormalization group, indicating the interdependence of the statistical fluctuations in complex phenomena. In our argument we employ scaling concepts borrowed from the Ising spin model of magnetic phase transitions, in order to be concrete in our discussion, but we avoid use of the detailed mathematics. We reserve most of the mathematics for the appendices.

### **1. Introduction**

The rhythmic changes of color in oscillating chemical reactions, the biorhythms that order human life, including the beating of the heart, and the changes in phases among physical phenomena are all consequences of the organization of countless unseen processes without which the observed effect would be absent. In the oscillating chemical reaction the nonlinear couplings among several of the reactants and products of the reaction (out of a large number of possible couplings) provides the feedback

mechanism necessary for the concentration of the chemical to vary harmonically in time. Similarly, the stability of the heartbeat appears to be due, at least in part, to the nonlinear couplings of the multiple cardiac pacemakers. In both of these examples the underlying complexity, the individual chemical reactions in the first instance and the electro-chemical degrees of freedom in the second, are entrained by a scaling process that acts across the dynamical ranges of each of the systems. Our physical understanding of such hierarchical coupling became clear with the development of renormalization group theory which, through the use of coarse-graining, demonstrated how the microscopic, short-range detailed interactions of a physical process are not important for the determination of the macroscopic, long-range patterns. This view enabled us to understand how physical systems as different as water and metal share certain fundamental properties in so far as transitions between phases are concerned.

Herein we shall sketch the development of our theoretical understanding of complex physical systems, starting with the Greek philosophers and atomic theory. We start so far back because the basic picture of the physical world has not changed very much in 2500 years. It is true that we now have formal mathematical theories that are able to make testable predictions regarding the behavior of the world, but as we shall see, those predictions require a great deal of theoretical background in order to properly interpret them. That is to say, predictions are not given in terms of an anticipated, definite, unambiguous outcome of an experiment, such as the result of a horse race. Rather, predictions are more likely to take the form of a weather report, with a given probability for rain. Why we settle for this latter type of prediction over the former is part of the story.

Let us begin by noting that all the physical and chemical laws that play important roles in the physical and life sciences are statistical in nature. Consider the simple example of a gas under pressure at a given temperature in a container. If the container confines  $N$  molecules of gas then at any moment in time the relations among the thermodynamic variables given by Boyle's law could be tested and would be found to be inaccurate by departures of order  $\sqrt{N}$ . According to Schrödinger, one of the founders of quantum mechanics, the  $\sqrt{N}$ -rule concerns the degree of inaccuracy to be expected in any physical law. The simple fact that the descriptive paradigm of natural order, the laws of physics, are inaccurate within a probable error of order  $1/\sqrt{N}$ , where  $N$  is the number of molecules involved in the phenomenon to which the law is applied, has been difficult for many outside the physical sciences to accept.

Buried in the  $\sqrt{N}$ -rule of Schrödinger is the legacy of Gauss and the law of errors, which we shall discuss in Section 2. The law of errors subsequently evolved into the Central Limit Theorem (CLT) and birthed the Gaussian (normal) distribution of probability. The application of these ideas to physical phenomena has led to the view that the macroscopic laws we observe in natural phenomena, such as Boyle's law, Ohm's law, Fick's law and so on, are all the consequence of the interaction of a large number of particles, and therefore can be described by means of statistical physics. Schrödinger, in his book "What is Life?", went on to say that because of the regularity in biology, in particular in heredity, that while still conforming to the laws of physics, biology might, in fact, have additional laws which had not up to that time been identified. The idea of formulating a new law in biology did not intimidate Schrödinger, since he, after all, had

been one of the architects of the quantum revolution in physics. This argument with only minor modifications can be applied with equal vigor to all manner of complex phenomena, including those in the social and medical sciences as well.

Much has changed in the fifty years since Schrödinger first passed along his insights regarding the application of physical law in biology. In particular the way we understand the statistics of complex phenomena in the context of the natural and social sciences as well as in life science is quite different from what it was then. Herein we do not propose a new law in Schrödinger's sense, but we do present in Section 3 evidence for the deviation from his  $\sqrt{N}$ -rule, which suggests that the statistical basis for Schrödinger's understanding of biology, or complexity in general, has changed. In the physical sciences we have often, in the past, used the words complex and complicated to mean the same thing. It is only recently that we have discovered that a process need not be complicated to generate complex phenomena, that is, complex phenomena need not have complex or complicated descriptions. In such cases the complicated nature of the phenomenon resides in the way the various scales contributing to the process are interconnected. This interconnectedness cannot be described by analytic functions, since the tying together of different scales prohibits the taking of derivatives necessary for the definition of such functions.

In the middle seventies Elliott Montroll and I adopted the statistical physics strategy to the study of complex phenomena, that strategy being to follow the possible evolution of large numbers of systems rather than one single system. In this way the erratic behavior of large numbers of single trajectories can be smoothed over to define probability densities. Even here, however, we found that there were some residual complications. The custom had been to replace a stochastic dynamical equation (a Langevin equation) describing the evolution of a single particle driven by a random force with a partial differential equation (the Fokker-Planck equation) to describe the evolution of the probability density for an ensemble of such particles. However, when the phenomenon is so erratic that the underlying process does not have finite variance, this approach breaks down and only an integral equation can be used to describe the evolution of the probability density. We discovered that the mathematician Paul Lévy had generalized the CLT in the 1930's to cover such situations and subsequently we found that this is the natural statistical distribution for describing many, if not most, complex phenomena in the natural, social and life sciences.

Lévy statistics, that we discuss in Section 3, naturally arise in the modeling of the evolution of phenomena that cannot be described by ordinary differential equations of motion; phenomena that leap and jump in unexpected ways to obtain food, that unpredictably twist and turn to avoid capture, and that suddenly change the direction of a gambit to evade checkmate. To understand these and other analogous processes we find that we must adopt a new type of modeling, one that is not in terms of differential equations of motion. It is clear that the fundamental elements of complex physical phenomena such as phase transitions, the deformation of plastics and the stress relaxation of polymers, satisfy Newton's laws. In these phenomena the evolution of individual particles are described by ordinary differential equations that control the dynamics of individual particle trajectories. It is equally clear that the connection between the fundamental laws of motion and the observed large-scale dynamics can not be made in any straightforward way. In fact the solution to one of these problem areas,

the renormalization group approach to phase transitions, seems to put the doctrine of reductionism into crisis.

In the twenty-five years the author has been interested in this problem area scientists have found that the ideas from renormalization group theory and the integral equations from probability theory dovetail as we discuss in Section 4. It is now clear that to understand certain classes of complex phenomena we must adopt a type of modeling that involves scaling, fractional differences and/or fractional derivatives. In Section 5 we summarize our discussion and draw some conclusions.

## **2. Normal Statistics**

The concept of complexity has undergone a remarkable change over the past half century. At the end of the nineteenth century the separation of phenomena into simple and complex was relatively straightforward. A simple system was one that could be described by one or a few variables and their equations of motion, given, for example, by Newton's laws of motion for material bodies or Maxwell's equations for the electromagnetic field. In such systems the initial conditions are specified and the final state is calculated (predicted) by solving the equations of motion. The predicted behavior of the system is then compared with the result of experiment and if the two agree with one another, within a pre-established degree of accuracy, the conclusion is that the simple model provides a faithful description of the phenomenon. Thus, simple physical systems have simple descriptions.

As more particles are added to the system there are more and more interactions, and the relative importance of any one interaction diminishes proportionately. There comes a point at which the properties of the system are no longer determined by the individual particle trajectories, but only by the averages over a large number of such trajectories. This is how the statistical picture of a physical phenomenon replaces the individual particle description. The single particle trajectory is replaced with the ensemble distribution function that describes an ensemble of single particle trajectories and the individual ordinary differential equations of motion fade into the partial differential equation of motion for the probability density. In this way the deterministic prediction of a definite future of a single particle that is characteristic of simple phenomena is replaced with a predicted collection of possible futures for an ensemble of particles that is characteristic of "complex" phenomena.

This is how statistical physics first emerged from classical dynamics in the hands of Maxwell working in England and Boltzmann working in Germany in the nineteenth century. Maxwell introduced the probability density into physics as a way of smoothing the complexity of the individual particle dynamics in gases, while Boltzmann did the same using the concept of entropy. The two methods came together at the turn of the century in the work of the American, Gibbs, who showed how to express the entropy of a system in terms of the probability density. We shall not review the concept of entropy here, except to note that the second law of thermodynamics can be expressed through the requirement that the entropy increases in every thermally isolated system. The monotonically increasing entropy is a measure of disorder and is not necessarily a measure of complexity, although we argue elsewhere in this encyclopedia that such measures can be constructed from the entropy.

This statistical picture of reality actually began with the atomic hypothesis of Democritus. He argued, 2500 years ago, that all matter was made up of microscopic, indestructible, bits of matter called atoms moving along deterministic trajectories in predominately empty space. Most of the time the particles move in straight lines, but on occasion two particles collide, changing their speeds and directions of motion. Although completely deterministic, there is such a large number of these scattering events (the number of two-particle scatterings increases as the square of the number of particles) that we can not keep track of them. Thus, to our limited human minds, the motion of the particles in a gas appears to be random; in this context randomness is a completely subjective concept. This is also the Maxwell-Boltzmann picture of the kinetic theory of gases. There are, of course, some refinements of the nineteenth century version over the original, with, for example, the probability density being expressed in terms of the kinetic energies of the individual particles. The essential ideas of Democritus and those of Maxwell and Boltzmann are, however, the same.

Somewhat later, another Greek, Epicurus, saw things a little differently. Rather than only the scattering events described by deterministic dynamics, and therefore in principle reversible in time, Epicurus believed in indeterminate events. He argued that in order to realize such non-deterministic processes as free will, there must be collisions among the atoms that are uncaused and are therefore objectively random. This fundamental randomness leads to an asymmetry in time, since the occasion of these events can not be anticipated, so that life can unfold in one direction in an unpredictable way and free will is again free.

Statistical physics has endorsed the view of Democritus with only a few dissenting voices, among the them the Noble Laureate, Ilya Prigogine, who, along with a few others, has in the past few decades put forward theories of microscopic irreversibility. Thus, we have both the views of Democritus and Epicurus represented in modern statistical physics. Herein we shall be primarily concerned with the dominant, that is to say the reversible, deterministic, view, since most of what we understand about the hierarchical structure of the physical world is based on this view. Thus, we are not concerned with the fundamental question of the direction of time, only with the vast number of scattering events making up a typical physical process.

Suppose we take a large number,  $N$  say, of measurements of a complex process, or sample a continuous time series  $N$  times at equally spaced time intervals, in either case we obtain the data set  $\{X_j\}$ ,  $j = 1, 2, \dots, N$ . The question now arises as to how best to characterize this set of  $N$  data points. The measure that has been used is that of the arithmetic average expressed as the summation over all the data points divided by the number of data points

$$X(N) = \sum_{j=1}^N X_j, \text{ and } \bar{X} = \frac{1}{N} X(N). \quad (1)$$

The overbar denotes the arithmetic averaging procedure and the quantity  $X(N)$  can be viewed as the displacement of a random walker after taking  $N$  steps of sizes  $X_1, X_2, \dots, X_N$ . If each of the individual step sizes are of unit magnitude, the variation

in the size of the displacement is  $\sqrt{N}$ , so that the variation in the average value is given by  $1/\sqrt{N}$ . Gauss was the first scientist to argue that the average is the best representation of a large data set, particularly a data set in which the variation in the measurement appears to have no structure or pattern. All thermodynamical variables had been thought to be averages of this kind over repeated measurements of a physical observable.

The next question is how reliable the mean is as a way to characterize the data. For example, a student with a C average obtained from all courses receiving a grade of C, is quite different from a student with a C average that receives half A's and half F's in the course work. The first student is truly average, knowing a minimum amount about all the required material. The second student, on the other hand, strongly oscillates between flashes of brilliance and either being completely uninterested or completely incompetent. In other words the mean only tells part of the story, with the variance and other, higher, moments of the data set containing additional information. The variance of the data set is calculated using the arithmetic average

$$\text{Var}X = \overline{X^2} - \bar{X}^2 = \frac{1}{N} \sum_{j=1}^N (X_j - \bar{X})^2 \quad (2)$$

from which we see that the variance is a measure of the magnitude of the variation in the data relative to the mean. The greater the variance, the greater is the degree of variation about the mean value, and the less reliable the mean becomes as a representative of the data set.

Suppose the data set is the local temperature, pressure or any other local thermodynamical property of the atmosphere. Each measurement involves the properties of  $N$  particles and therefore the measurement contains fluctuations or errors on the order  $\sqrt{N}$ . The  $N$  used here does not refer to all the particles in the atmosphere, but to those in the immediate vicinity of the measurement. This  $N$  is of the order  $10^{23}$ , which is certainly large enough for statistics. Thus, Schrödinger's  $\sqrt{N}$ -rule indicates that it is not possible to find the "true" value of a thermodynamic quantity from a single measurement. The instantaneous value of the thermodynamical variable has no meaning, only the ensemble of measurements has meaning - the mean value. In the theory of error of measurements, a discipline invented by Gauss, the probability density describing the statistical properties in a data set was found to be the bell-shaped distribution of Gauss. The Gaussian distribution is able to characterize the entire data set in terms of two quantities, the mean and the variance.

The mathematicians were able to generalize these ideas out of their physical context and to identify the basic conditions underwhich a normal (Gaussian) distribution results. In Appendix A we sketch how the sum of random variables given by (1) satisfies the conditions for the CLT. In particular, we see that all distributions for which the mean square step length is finite, are attracted to the Gaussian distribution, which is to say, the statistics of the properly scaled random walk variable  $X(N)/\sqrt{N}$  becomes normal as the number of steps becomes infinitely large. This applies for all distributions of the step sizes that decrease more rapidly than an inverse cube of the step length. These

properties are summarized in the conditions: (1) the distribution of the random steps must be sufficiently narrow that the second moment is finite; (2) the random variable must not have long-range correlations; (3) the statistics of each of the random steps must be the same and (4) there must be a sufficient number of steps. This is the content of the CLT in its usual form. However, Poincaré said of the law of errors: "All the world believes it firmly, because the mathematicians imagine that it is a fact of observation and the observers that it is a theorem of mathematics." He was, of course, alluding to the fact that the measurements of a given phenomenon does not necessarily satisfy the underlying assumptions necessary for the law of errors and one must examine each process separately.

This is how the matter stood at the turn of the century. The probability that a random variable  $X$  lies in the phase space interval  $(x, x + dx)$  is given by  $P(x)dx$  where the probability density is given by

$$P(x, \bar{X}, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x - \bar{X})^2}{4\sigma^2}\right] \quad (3)$$

where  $\sigma^2$  is shorthand for the variance (2), and the parametric dependence of the probability density is explicitly indicated. If  $X$  is interpreted as the single-particle momentum of the molecules in a gas and the variance is expressed in terms of the temperature,  $T$ , of the gas,  $\sigma^2 = mkT/2$ , where  $m$  is the mass of the molecule and  $k$  is Boltzmann's constant, then (3) is the Maxwell-Boltzmann distribution.

Thus, normal statistics is a consequence of the interactions of the many elements in a system being short-ranged. This is the domain of equilibrium statistical mechanics, where the system is spatially featureless and all the physical properties of the phenomenon are determined by means of averages over ensemble distribution functions of the form (3) or other equally benign distributions. We know this description is adequate for the thermodynamical properties of a gas, the conduction of electrons in normal metals, and normal diffusion. The key feature of such processes is that the time scale for the microscopic interaction is very much smaller than the typical time scale for macroscopic observables. It is this time scale separation that gives rise to the CLT in its classical form. However, what happens when the interactions are no longer short-range or the system is no longer in equilibrium? This is the case when the separation of time scales does not occur and the interaction time of microscopic processes may overlap with those of macroscopic processes.

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## Biographical Sketch

**Bruce J. West** is a leading thinker in the applications of nonlinear dynamics and in the sciences of complexity. He received his PhD in physics from the University of Rochester in 1970. In 1978 he was a founding member of the La Jolla Institute (LJI), a private non-profit research corporation. In 1979 he became the first Associate Director of the Center for Studies of Nonlinear Dynamics (CSND), a division of the LJI. This was the first of the centers devoted to the study of nonlinear phenomena. Dr. West became the Director of the other division of LJI, the Division for Applied Nonlinear Problems (DANP). From 1989 to 1999 he was a Professor of Physics at the University of North Texas where he established the Center for Nonlinear Science on the University of North Texas campus. In June of 1999 he left the university and joined the Army Research Office.

Dr. West's research interests are in the mathematical and physical modeling of complex phenomena, from physical processes whose evolution cannot be described by differential equations of motion, to biomedical phenomena that are so complex that we have little idea of how to mathematically describe them.

Dr. West is the author of several books, including *Fractal Physiology and Chaos in Medicine* (1990), *The Lure of Modern Science: Fractal Thinking* (1995), *Physiology, Promiscuity and Prophecy at the End of the Millenium: A Tale of Tails* (1999).