EQUILIBRIUM AND STABILITY ANALYSIS

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Contents

Summary

In this chapter some dynamical systems concepts of relevance to Systems Dynamics are discussed. First of all, the concept of equilibrium of a dynamical system is associated with the attractors of the system, which characterize the long-term behavior. Then it is emphasized that a nonlinear dynamical system can display several attractors. This leads to the need for an overall perspective on all such long-term behavior. This global perspective supplies a valuable guide to System Dynamics simulation. Concepts and tools taken from bifurcation theory are of a great help in this task. The archetypes supplied by this theory permit us to organize the different aspects of long-term behavior in a single framework. The chapter closes with the application of the time-scale decomposition to apply these analysis tools to systems of higher dimension.

1. Introduction

System Dynamics is both a modeling and a simulation methodology. As a modeling technique it focuses on the complex structure of the aspect of reality being modeled. That complexity is due to multiloop feedback substructures, nonlinearities and delays. The consideration of these characteristics leads to mathematical models that are quite difficult to deal with using analytical tools. In this way some dilemma is produced between mathematical analysis and computer modeling. Researchers in System Dynamics have chosen to do the analysis with the use of the computer. Nevertheless, mathematical analysis yields benefits for System Dynamics that should not be disregarded as they can help to better understand the behavior of a model. If a mathematical language is used, the consequences of that usage ought to be fully maintained. A System Dynamics model is a mathematical object and, as such, it deserves mathematical analysis.

2. Equilibrium Points

Any System Dynamics model is a mathematical object known as dynamical system, that is, it can be formulated as

$$\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is a vector of the state of the system (usually called *levels* in System Dynamics), $\dot{x}$ represents the derivative of x with respect to time and f is a possibly nonlinear function. Normally, the function f is highly involved as it represents the whole model, but, at least conceptually, a model can be thought as a mathematical object of the form (1).

The equilibrium points (or simply the equilibria) of a model like (1) are defined as the values x_e of x for which if, at a specific time t_0, $x = x_e$ then x will remain unchanged for all $t > t_0$. This implies that all the model variables will remain constant and, therefore, $\dot{x} = 0$. This condition can be used to obtain the equilibrium points of a system.

As will be seen later, the number and values of the equilibria of a system play an important role in the behavior of that system. An equilibrium point x_e is said to be stable when initial conditions close to that point produce trajectories (time evolutions of x) which approach the equilibrium. On the contrary, if these trajectories move away from x_e, the equilibrium is unstable.

Any dynamical system may have no, one or several equilibrium points, each of which may either be stable or unstable. A plain mechanical example is a simple pendulum. If the pendulum is at the downward position, the system will remain there forever, and thus, the downward position is an equilibrium point. Obviously, this equilibrium is stable (if friction is considered). Likewise, it is easy to see that the upward position is an unstable equilibrium.

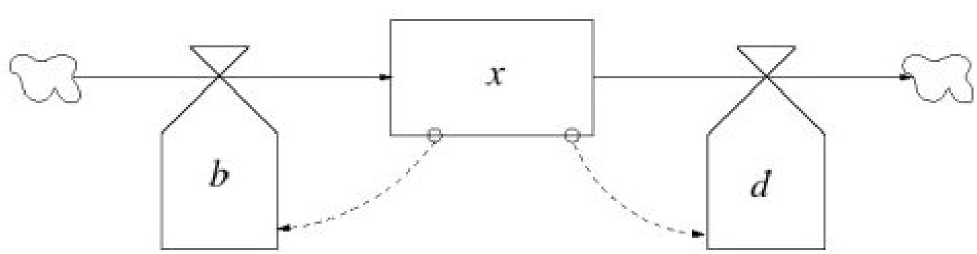

Figure 1: Flow diagram of a population model.

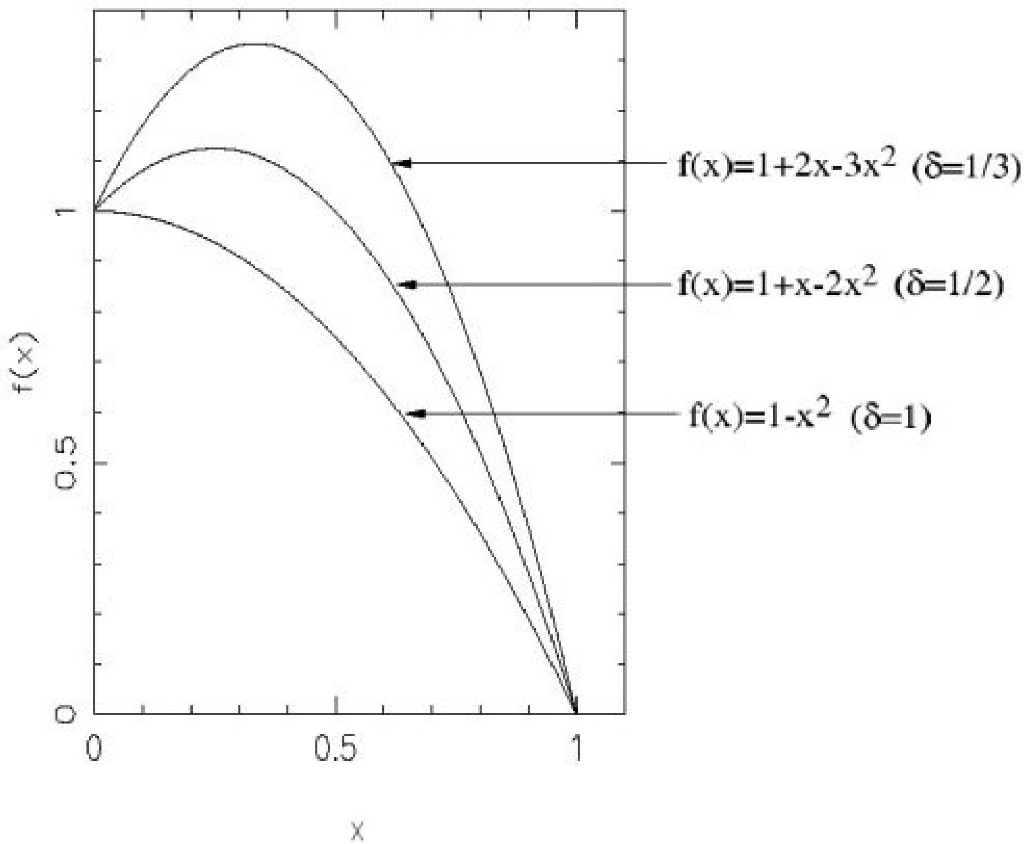

Figure 2: Graph of $\tilde{f}(x)$ for $\delta = \frac{1}{3}$, $\delta = \frac{1}{2}$ and $\delta = 1$.

Another example, more related to System Dynamics, and which will be used below is the following (see Figure 1). Consider a population model as follows:

$$\dot{x} = b(x) - d(x), \quad x > 0, \qquad (2)$$

where x represents the number of inhabitants in a certain region; $b(x)$ stands for the birth rate and $d(x)$ is the death rate. Assume that the death rate is proportional to the population, i.e. $d(x) = mx$. For the birth rate, a nonlinear dependence on the population, namely $b(x) = nxf(x)$, is assumed, where the function f is supposed to have a qualitative "hump shape", increasing for low values of x and later decreasing. The reason for this is that small populations reproduce with more difficulty and very large populations may have resource limitations. Therefore, there will be a higher birth rate for medium-sized populations. For example, function f could be of the form:

$$\tilde{f}(x) = 1 + \left(\frac{1-\delta}{\delta}\right)x - \frac{x^2}{\delta}, \quad \delta \in (0,1],$$

which is represented in Figure 2 for several values of δ. In Figure 3 a more detailed flow diagram of the model is presented.

This model usually shows a logistic growth as is shown in Figure 2. System Dynamics includes the archetype known as *limits to growth*, which shows this kind of growth. This methodology allows an interpretation in which this process is the result of the interaction of a positive feedback loop, which is responsible for the initial growth, and a negative feedback loop, to which the final stabilization can be attributed. Throughout the process, there is a change in loop dominance. So, positive feedback loop domination in the initial phase gives way to the negative feedback loop in the final phase.

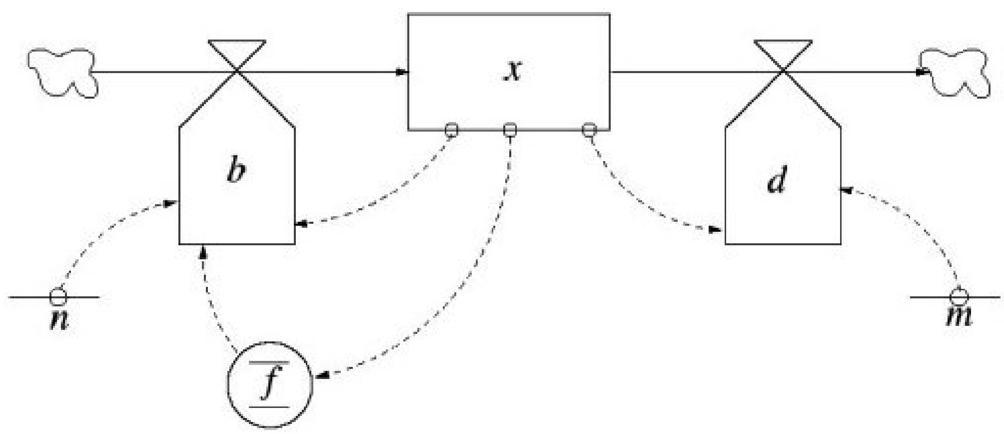

Figure 3: System dynamics stock-flow diagram of a specific population model.

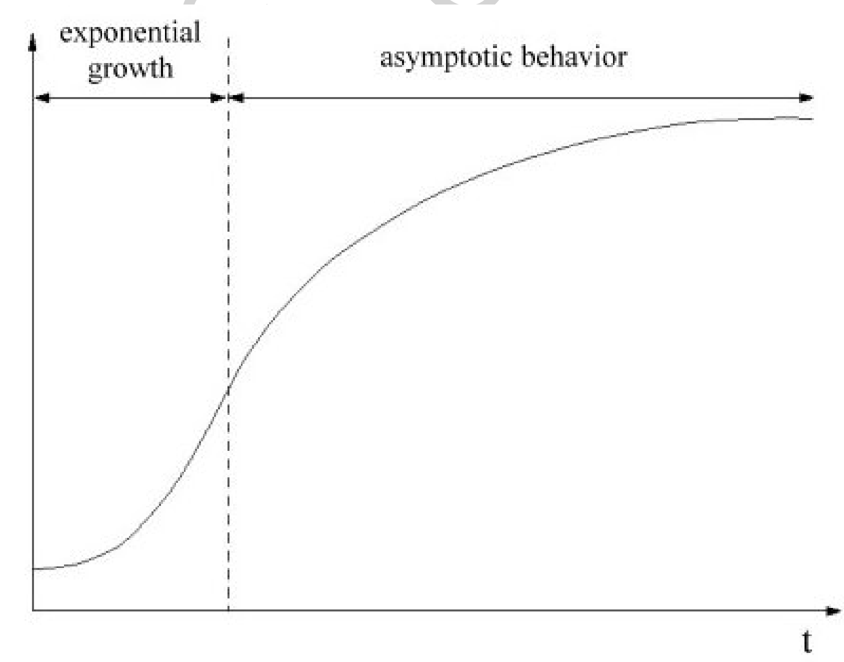

Figure 4: Logistic growth curve.

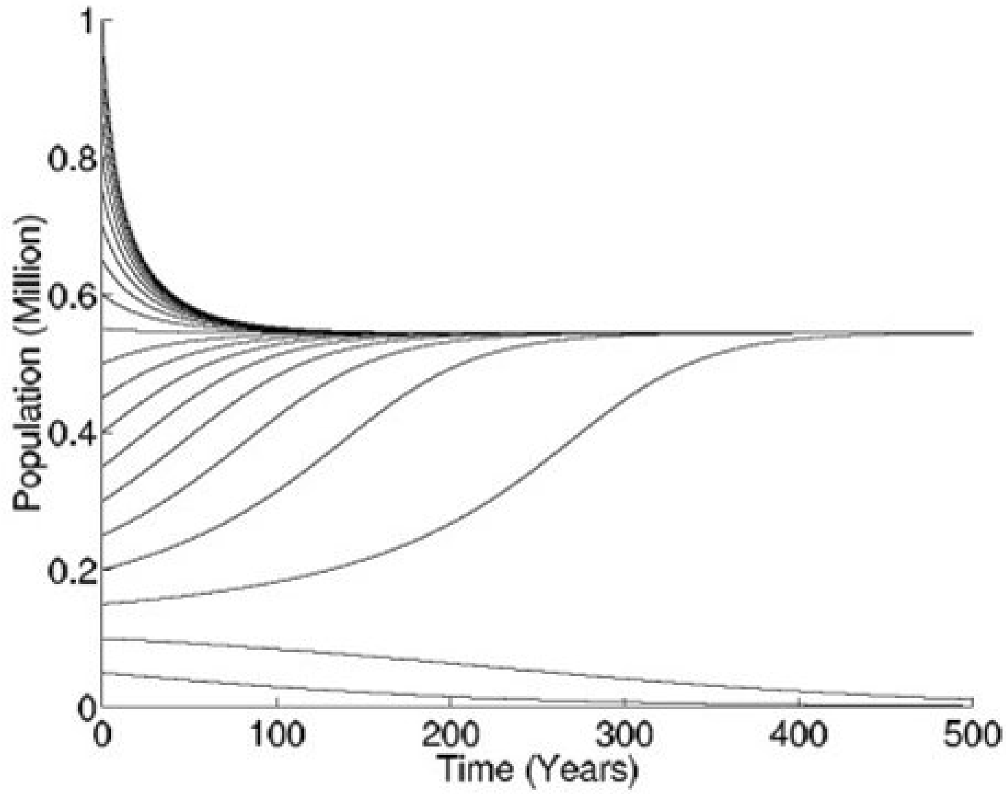

Figure 5: Evolution of *x* for several values of *x*(0)

This change of loop dominance is associated with nonlinearities in the structure of the model. These nonlinearities, under certain circumstances (for certain parameter values of the model), can cause a bimodal behavior pattern in which there are two attractors, which correspond to long-term behavior defined, in one case, by logistic growth of the population and, in the other, by its decay and extinction.

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Biographical Sketches

Javier Aracil was born in Alcoy, Spain, in 1941. He is currently Professor of Automatic Control at the Department of Automatic Control and Director of the Institute of Automatic Control and Robotics, both at the University of Seville (Spain). His research interest are in the areas of theory and philosophy of dynamical systems modeling and control, with emphasis on the application of qualitative methods (bifurcations, qualitative change, chaos,…) to system dynamics models and to control systems. He received the 1986 Jay W. Forrester Award for his contributions to this research area. He has also been awarded by the 1991 Premio Andalucía de Investigación. He is the author (or co-author) of six books and of numerous papers, and a Member of the Spanish Academy of Engineering.

Francisco Gordillo was born in Zafra, Spain, in 1964. He received the Ingeniero Industrial and the Doctor Ingeniero Industrial degrees, from the Universidad de Sevilla, Spain in 1988 and 1994, respectively. Since 1989 he has been with the Department of Automatic Control of the Escuela Superior de Ingenieros of the Universidad de Sevilla where he is currently an Associate Professor. He is coauthor of *Dinámica de Sistemas* (Madrid: Alianza Editorial, 1997), co-editor of *Stability Issues in Fuzzy Control* (Berlin: Physica-Verlag, 2000) and author or co-author of more than 30 publications including book chapters, papers in journals and conference proceedings. His research interest are in the area of dynamical systems modeling and control, with emphasis on the application of qualitative methods (bifurcations, qualitative change,…) to system dynamics models and to control systems.