DESCRIBING FUNCTION METHOD

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Keywords: Sinusoidal describing function, Fourier series, nonlinearity, limit cycle, stability, accuracy, distortion, frequency response, other describing functions.

Contents

Summary

This section is primarily concerned with developing the background of the describing function for a single sinusoidal signal and showing how it can be used in the analysis and possibly the design of a nonlinear feedback system. After the definition of the describing function, its value is obtained for several specific nonlinear characteristics and then it is shown how the information can be used to explore the possibility of limit cycles in a nonlinear feedback loop. It is shown how the stability of any limit cycles may be ascertained and two examples of use of the DF in control systems problems are given. Uses of the DF for evaluating the closed loop frequency response and for designing compensators to eliminate limit cycles are discussed. The latter part of the presentation discusses describing functions for other signals, including those consisting of more than one component, and their possible uses in studying some aspects of feedback loop analysis and design.

1. Introduction

The describing function, which will be abbreviated DF, method was developed simultaneously in several countries during the 1940s. Engineers found that control systems which were being used in many applications, for example gun pointing and antenna control, could exhibit limit cycles under certain conditions rather than move to a static equilibrium. They realized this instability was due to nonlinearities, such as backlash in the gears of the control system, and they wished to obtain a design method which could ensure the resulting systems were free from limit cycle operation. They observed that when limit cycles occurred the waveforms at the system output were often approximately sinusoidal and this indicated to them a possible analytical approach, namely to assume that the signal at the input to the nonlinear element in the loop was a sinusoid. Since then there have been many developments in terms of both using the DF concept for other types of signals and the problems, or phenomena, which they can be used to study. More will be said on these aspects later but we begin by considering the initial problem of investigating the possibility of a limit cycle in a feedback system using the DF or S (sinusoidal) DF as it is often named.

Consider the autonomous feedback system shown in Figure 1 containing a single static nonlinearity $n(x)$ and linear dynamics given by the transfer function $G(s) = G_c(s)G_1(s)$. If a limit cycle exists in the autonomous system with the output $c(t)$ approximately sinusoidal, then the input $x(t)$ to the nonlinearity might also be expected to be sinusoidal. If this assumption is made the fundamental output of the nonlinearity can be calculated and conditions for the sinusoidal self-oscillation found, if the higher harmonics generated at the nonlinearity output are neglected. This is the concept of harmonic balance, in this case balancing the first harmonic only, which had previously been used by Physicists to investigate such aspects as the generation of oscillations in electronic circuits. The DF of a nonlinearity was therefore defined as its gain to a sinusoid, that is the ratio of the fundamental of the output to the amplitude of the sinusoidal input. Since the output fundamental may not be in phase with the sinusoidal input the DF may be complex.

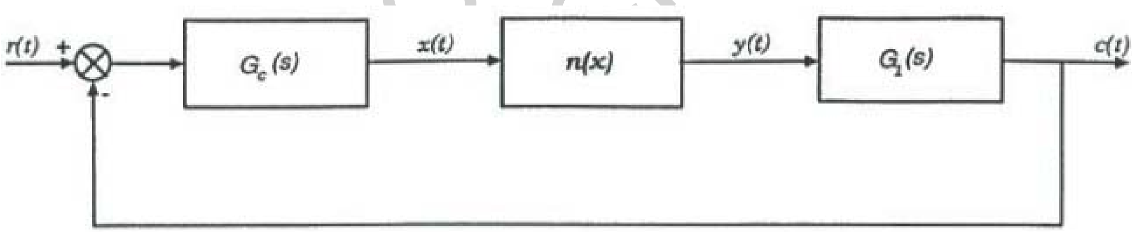

Figure 1: A simple nonlinear feedback system

2. The Sinusoidal Describing Function

Assume that in Figure 1 $x(t) = a\ \cos\theta$, where $\theta = \omega t$ and $n(x)$ is a symmetrical odd nonlinearity, then the output $y(t)$ will be given by the Fourier series.

$$y(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

where $a_n = b_n = 0$ for n even, and in particular

$$a_1 = (1/\pi)\int_0^{2\pi} y(\theta)\ \cos\ \theta\ d\theta \qquad (1)$$

and

$$b_1 = (1/\pi)\int_0^{2\pi} y(\theta)\ \sin\theta\ d\theta \qquad (2)$$

The fundamental output from the nonlinearity is $a_1\cos\theta + b_1\sin\theta$, so that the DF is given by

$$N(a) = (a_1 - jb_1)/a$$

which may be written

$$N(a) = N_{\mathrm{p}}(a) + jN_{\mathrm{q}}(a)$$

where

$$N_{\mathrm{p}}(a) = a_1/a \quad \text{and} \quad N_{\mathrm{q}}(a) = -b_1/a$$

Alternatively, in polar co-ordinates

$$N(a) = M(a)e^{j\psi(a)}$$

where

$$M(a) = (a_1^2 + b_1^2)^{1/2}/a$$

and

$$\psi(a) = -\tan^{-1}(b_1/a_1).$$

If $n(x)$ is single valued it is easily shown that $b_1 = 0$ and

$$a_1 = (4/\pi)\int_0^{\pi/2} y(\theta)\cos\theta d\theta \qquad (3)$$

giving

$$N(a) = (a_1/a) = (4/a\pi)\int_0^{\pi/2} y(\theta)\cos\theta d\theta \qquad (4)$$

Although Eqs. (1), (2) are an obvious approach to the evaluation of the fundamental output of a nonlinearity, they are somewhat indirect, in that one must first determine the output waveform $y(\theta)$ from the known nonlinear characteristic and sinusoidal input waveform. This is avoided if the substitution $\theta = \cos^{-1}(x/a)$ is made; in which case, after some simple manipulations, it can be shown that

$$a_1 = (4/a)\int_0^a x n_{\mathrm{p}}(x)\, p(x)\, dx \tag{5}$$

$$b_1 = (4/a\pi)\int_0^a n_{\mathrm{q}}(x)\, dx \tag{6}$$

The function $p(x)$ is the amplitude probability density function of the input sinusoidal signal and is given by

$$p(x) = (1/\pi)(a^2 - x^2)^{-1/2}$$

and the nonlinear characteristics $n_{\mathrm{p}}(x)$ and $n_{\mathrm{q}}(x)$, called the in-phase and quadrature nonlinearities, are defined by

$$n_{\mathrm{p}}(x) = [n_1(x) + n_2(x)]/2$$

and

$$n_{\mathrm{q}}(x) = [n_2(x) - n_1(x)]/2$$

where $n_1(x)$ and $n_2(x)$ are the portions of a double valued characteristic traversed by the input for $\dot{x} > 0$ and $\dot{x} < 0$ respectively. For a single-valued characteristic, $n_1(x) = n_2(x)$, so that $n_{\mathrm{p}}(x) = n(x)$ and $n_{\mathrm{q}}(x) = 0$. Also integrating Eq. (5) by parts gives

$$a_1 = (4/\pi)\, n(0^+) + (4/a\pi)\int_0^a n'(x)(a^2 - x^2)^{1/2}\, dx \tag{7}$$

where $n'(x) = dn(x)/dx$ and $n(0^+) = \lim_{\varepsilon \to 0} n(\varepsilon)$; a useful expression for obtaining DFs for linear segmented characteristics.

An additional advantage of using Eqs. (5) and (6) is that they easily yield proofs of some interesting properties of the DF for symmetrical odd nonlinearities. These include the following:

1. For a double-valued nonlinearity the quadrature component $N_{\mathrm{q}}(a)$ is proportional to the area of the nonlinearity loop, that is: $N_{\mathrm{q}}(a) = -(1/a^2\pi)$ (area of nonlinearity loop)
2. For two single-valued nonlinearities $n_\alpha(x)$ and $n_\beta(x)$, with $n_\alpha(x) < n_\beta(x)$ for all $0 < x < b$, then $N_\alpha(a) < N_\beta(a)$ for input amplitudes a less than b
3. For the sector bounded single-valued nonlinearity that is $k_1 x < n(x) < k_2(x)$ for all $0 < x < b$ then $k_1 < N(a) < k_2$ for input

amplitudes a less than b. This is the sector property of the DF and it also applies for a double-valued nonlinearity if $N(a)$ is replaced by $M(a)$.

When the nonlinearity is single valued, it also follows directly from the properties of Fourier series that the DF, $N(a)$, may also be defined as:

1. The variable gain, K, having the same sinusoidal input as the nonlinearity, which minimizes the mean squared value of the error between the output from the nonlinearity and that from the variable gain
2. The covariance of the input sinusoid and the nonlinearity output divided by the variance of the input

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Bibliography

Astrom K. J. and Hagglund T. (1984). Automatic tuning of simple regulators with specifications on phase and amplitude margins. *Automatica*, Vol. 20, No. 5, pp 645-651 [The first paper to suggest using a relay to replace the proportional gain in the Ziegler Nichols loop cycling method. The advent of microprocessor controllers made the approach practical. Limit cycle result obtained using the known describing function method.]

Atherton, D. P. (1975). *Nonlinear Control Engineering: Describing Function Analysis and Design.* London: Van Nostrand Reinhold, 1975 [A book with probably the most detailed coverage of classical approaches for nonlinear control systems, with major concentration on describing function methods.]

Atherton D. P. (1981). *Stability of Nonlinear Systems*. Research Studies Press, John Wiley. [Reviews classical methods for stability of nonlinear systems with emphasis on limit cycle determination, including exact methods for relay systems.]

Balasubramanian, R. (1981). Stability of limit cycles in feedback systems containing a relay. *IEE Proc. D*, No. 1, pp 24-29 [The paper discusses the stability of limit cycles in relay systems, the conjecture suggested when discontinuities occur at switching has since been proven incorrect.]

Choudhury, S. K. and Atherton, D. P. (1974). Limit cycles in high-order nonlinear systems. *Proc. Inst. Electr. Eng.* 121, pp 717-724 [The paper examines the stability of limit cycles in systems where the DF method gives more than one solution.]

Cook P. A. (1994). *Nonlinear Dynamical Systems.* London: Prentice-Hall International, 2nd ed [A book primarily covering classical control with interesting material on chaos, stability and relay systems.]

Cosgriff, R. L. (1958). *Nonlinear Control Systems.* New York: McGraw Hill [One of the early books devoted to extensions of classical control methods for the study of nonlinear control systems.]

Gelb, A. and Vander Velde, W. E. (1996). *Multiple-Input Describing Functions and Nonlinear System Design.* New York: McGraw Hill [The first textbook to give extensive coverage to describing function methods and their use in feedback control.]

Gibson, J. E. (1963). *Nonlinear Automatic Control*. New York: McGraw Hill [A slightly more extensive coverage of nonlinear classical control than earlier books.]

Graham, D. and McRuer, D. (1961). *Analysis of Nonlinear Control Systems*. New York: Wiley [A good textbook on classical methods for nonlinear control.]

McNamara O. P. and Atherton D. P. (1987). Limit Cycle Prediction in Free Structured Nonlinear Systems, *IFAC Congress, Munich*, Vol. 8, pp 23-28 [A paper describing a harmonic balance computational method for obtaining limit cycles starting from a describing function estimate.]

Singh Y. P. (1965). Graphical Method for Finding the Closed loop Frequency Response of Nonlinear Feedback Control Systems. *Proc. IEE.*, 112, pp 2167-70 [A graphical approach for the study of the closed loop frequency response of a nonlinear feedback loop and the possibility of jump resonance.]

Slotine J. J. E. and Li W. (1991). *Applied Nonlinear Control.* Prentice-Hall International [Reasonable coverage of classical control methods plus coverage of state space methods for sliding motion, feedback linearization and adaptive control.]

Taylor, J. H. (1980). Applications of a general limit cycle analysis method for multivariable systems, in Ramnath, R. V., Hedrick, J. K. and Paynter, H. M (eds) *Nonlinear System Analysis and Synthesis*. New York: ASME, Vol. 2 [Presents an approach based on describing functions for obtaining limit cycles in nonlinear multivariable systems, with some interesting applications.]

Taylor, J. H. and Astrom, K. J. (1986). *A Nonlinear PID Autotuning Algorithm.* Seattle WA: ACC, pp 1-6 [Considers autotuning of PID controllers for plants with nonlinear dynamics.]

Thaler, G. J. and Pastel, M. P. (1962). *Analysis and Design of Nonlinear Feedback Control Systems*. New York: McGraw Hill [Good coverage of nonlinear classical control methods typical of books in the 1955-1965 period.]

West, J. C., Douce, J. L. and Naylor, R. (1954). The effects of some nonlinear elements on the transient performance of a simple r.p.c. system possessing torque limitation. *Proc. IEE*, 101, pp 156-165 [An early paper using the phase plane method to study the effect of nonlinearity approximated by linear segments.]

West J.C., Douce J. L., and Livesley R. K. (1956). The dual input describing function and its use in the analysis of nonlinear feedback systems. *Proc. IEE*, Part B, Vol. 103 pp 463-474 [The first paper to present a dual input describing function for related sinusoidal inputs and to show how it could be used to predict aspects of nonlinear system behavior.]

Ziegler J. G. and Nichols N. B. (1942). Optimum settings for automatic controllers. *Trans. ASME*, Vol. 64, pp 759-768. [The original paper suggesting how the parameters of PID controllers might be chosen based on one of two simple tests. The first using an open loop plant step response, the second using a closed loop test with proportional control and the gain adjusted to produce oscillations.]

Biographical Sketch

Professor D P Atherton was born in Bradford, England on 21 April 1934. He has a B.Eng from Sheffield University and Ph.D and D.Sc from Manchester University. He taught at Manchester University, and McMaster University and the University of New Brunswick in Canada before taking up the appointment of Professor of Control Engineering at the University of Sussex in 1980, where he currently has a part-time appointment. He has served on committees of the Science and Engineering Research Council, as President of the Institute of Measurement and Control in 1990 and President of the Control Systems Society of the Institute of Electrical and Electronic Engineers, USA in 1995, and also served for six years on the International Federation of Automatic Control (IFAC) Council. His major research interests are in nonlinear control theory, computer aided control system design, simulation and target tracking. He has written three books, one of which is jointly authored, and published over 300 papers.