DESIGN TECHNIQUES FOR TIME-VARYING SYSTEMS

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Contents

Summary

Model descriptions of time-varying systems, both in the time and frequency domains, are presented. Various stabilization techniques, differing as to the assumptions required on the plant are then discussed. It is shown that many design schemes used for time-invariant systems have straightforward extensions to the time-varying case.

5 Introduction

In this article, we consider the design of controllers for linear, time-varying systems. Interest in the analysis of differential equations with time-varying coefficients dates to the earliest days of control analysis. Early pioneers such as Floquet and Lyapunov considered basic issues relating to the stability and reduction of these systems. Starting in the 1950s, there was considerable interest in extending to time-varying systems the integral transform techniques that were so useful in the study and design on linear time-invariant systems. In the 1960s, these efforts were eclipsed by the time-domain, state-space approach that swept the control systems community. This led to the development of many of the fundamental issues related to time-varying systems. In the 1980s, problems related to the robustness of feedback control systems spurred a renewed interest in frequency domain methods. This, once again, generated interest in input-output and frequency domain analysis and synthesis of linear time-varying control systems.

Time-varying models for systems typically arise in one of two ways. Consider the system described by the vector differential equation

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{1}$$

where $\boldsymbol{x}(t)$ is the state and $\boldsymbol{v}(t)$ is the control at time t. Assume that $\bar{\boldsymbol{v}}$ is an open-loop control and $\bar{\boldsymbol{x}}$ is a reference trajectory which, together, satisfy Eq. (1). Defining the differences

$$\delta(t) = \boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t), \quad \text{and} \quad \boldsymbol{v}(t) = \boldsymbol{u}(t) - \bar{\boldsymbol{u}}(t),$$

and ignoring higher order terms, these quantities satisfy the first order equation

$$\dot{\delta}(t) = \boldsymbol{A}(t)\delta(t) + \boldsymbol{B}(t)u(t)$$

Here $A(t) = f_x(\bar{\boldsymbol{x}}(t), \bar{\boldsymbol{v}}(t))$ and $B(t) = f_u(\bar{\boldsymbol{x}}(t), \bar{\boldsymbol{v}}(t))$ are the Jacobian matrices evaluated along the trajectory. Note that if the open-loop control is constant *and* the desired trajectory is a fixed point, then the resulting equation will be linear, time-invariant.

A second way in which time-varying models can arise is by considering parameter variations in a physical model over the time-horizon of a system. For example, consider the model for a simple harmonic oscillator

$$m\ddot{y}(t) + ky(t) = 0$$

where m corresponds to the mass and k is the force constant for the spring. If the mass and spring constant do not vary over time, this differential equation is that of a linear time-invariant system. Note that over short time-horizons, most systems can be described accurately by this type of time-invariant model. However, over longer periods, parameters in these time-invariant models will tend to exhibit variation. In this case, if either the mass of the oscillator, or the spring constant vary with time: that is, $m \equiv m(t)$ or $k \equiv k(t)$, then a linear, time-varying model will be required.

Typically, these parameter variations will fall under one of three classes:

1. Slow parameter drift. By “slow” we mean that the system parameters are

changing at a rate considerably slower than the dynamics of the system. These are the systems that will be primarily considered in this article.

2. "Jumps" or sudden variations. In this case, the parameters are usually assumed to be piecewise constant between the jumps. The times at which the parameters change may or may not be known *a priori*.
3. Fast variations. These are systems in which the time-variations form an integral part of the system. An example of this class of system is AM modulation. In this case, a signal $x(t)$ is modulated by a sinusoid $\sin(\omega_m t)$. This can be thought of as a time-varying gain $k(t)$ whose time-variation is usually faster than the bandwidth of the input signals.

The rest of this article is organized as follows. In Section 6, we introduce different model descriptions appropriate for time-varying systems. We present the state-space description as well as two general classes of input-output models. We also introduce the use of frequency domain description of time-varying systems.

In Section **Error! Reference source not found.**, we present some basic stabilization techniques that are appropriate for time-varying systems. Formulae for stabilizing controllers are given for both the case where the whole state is available to the designer as well as the more restrictive output feedback assumption.

One drawback of the schemes presented in Section **Error! Reference source not found.** is that the designer must have *a priori* knowledge of the time-variations of the systems. Two methods for over-coming this difficulty are considered in Section **Error! Reference source not found.**. This first requires that the system be slowly time-varying. A second approach that has come into prominence only recently is the control of linear *parameter*-varying systems.

We should note that most of the techniques that are used to design controllers for linear time-varying systems are modifications of their time-invariant counterparts. For this reason, our presentation will highlight the corresponding differences and modifications. The reader is advised to consult the time-invariant methods first.

6 Model Descriptions

In this section, we discuss different models of linear time-varying systems that are used in the literature.

6.1 State-Space Models

We begin by considering state-space descriptions of time-varying systems. For a more general discussion of state-space representations, see *Canonical State Space Representations and Feedback* .

Consider the finite-dimensional ordinary differential equation

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \end{aligned} \tag{2}$$

where $u(t) \in \mathbb{R}^m$ and $\boldsymbol{y}(t) \in \mathbb{R}^p$ are the input and output vectors respectively, and $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector. As we saw in Section 5, state-space models of linear time-varying systems can arise as linearizations of nonlinear systems when the linearization is done with respect to a nominal non-constant trajectory.

Given an arbitrary input $\boldsymbol{u}(t)$, it is possible to write a formula for the state and output satisfying

Eq. (2). To do this, we first need the *transition matrix* of the homogeneous part of Eq. (2). The transition matrix is defined by the *Peano-Baker series*:

$$\Phi(t,\sigma) = I + \int_\sigma^t \boldsymbol{A}(\tau_1)d\tau_1 + \int_\sigma^t \boldsymbol{A}(\tau_1) \int_\sigma^t \boldsymbol{A}(\tau_2)d\tau_2 d\tau_1 + \int_\sigma^t \boldsymbol{A}(\tau_1)\int_\sigma^{\tau_1} \boldsymbol{A}(\tau_3)d\tau_3 d\tau_3 d\tau_2 d\tau_1 + \cdots$$

which converges uniformly and absolutely for any t and τ. For arbitrary $A(t)$, it is not possible to simplify this expression to provide an analytical expression for the transition matrix. Two important exceptions are when the matrix is time-invariant, in which case $\boldsymbol{\Phi}(t, \sigma) = \exp[\boldsymbol{A}(t-\sigma)]$ or when the system's "$\boldsymbol{A}(t)$" matrix commutes with it integral:

$$\boldsymbol{A}(t)\int_\sigma^t \boldsymbol{A}(\tau)d\tau = \int_\sigma^t \boldsymbol{A}(\tau)d\tau \boldsymbol{A}(t), \qquad \text{for all } t \text{ and } \tau$$

In this case $\boldsymbol{\Phi}(t, \sigma) = \exp[\int_\sigma^t \boldsymbol{A}(\tau)\mathrm{d}\tau]$. Note that the transition matrix is always invertible and $\boldsymbol{\Phi}(t, \tau)^{-1} = \boldsymbol{\Phi}(\tau, t)$. Moreover, it obeys the composition property

$$\Phi(t,\tau) = \Phi(t,\sigma)\Phi(\sigma,\tau)$$

for all t, τ and σ.

Using the transition matrix, the solution to Eq. (2) consists of two components,

$$\begin{aligned} x(t) &= \underbrace{x_{\mathrm{zi}}(t)}_{\text{zero input}} + \underbrace{x_{\mathrm{zs}}(t)}_{\text{zero state}} \\ &= \Phi(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^t (t,\sigma)\boldsymbol{B}(\sigma)\boldsymbol{u}(\sigma)d\sigma \end{aligned}$$

and thus

$$\boldsymbol{y}(t) = \boldsymbol{D}(t)\boldsymbol{u}(t) + \int_{t_0}^t \boldsymbol{G}(t,\sigma)u(\sigma)d\sigma + \boldsymbol{C}(t)\Phi(t,t_0)\boldsymbol{x}_0 \tag{3}$$

where

$$\boldsymbol{G}(t,\sigma)=\boldsymbol{C}(t)\Phi(t,\sigma)\boldsymbol{B}(\sigma) \tag{4}$$

is the *weighting pattern* of the system, and this function is defined for all values of t and τ.

6.2 Input-Output Models

We will consider two other descriptions of linear time-varying systems, both based on the system's input-output behavior. To avoid carrying numerous indices we restrict our attention to single-input, single-output systems, though multivariable generalizations are straightforward.

6.2.1 Impulse Response

Our first description is analogous to the time-invariant impulse response. Suppose that an impulse $u(t) = \delta(t - \tau)$ is applied to a linear time-varying system at time τ (here $\delta(t)$ refers to the Dirac delta function). The corresponding output is

$$y(t)=g(t,\tau), \qquad t\geq\tau \tag{5}$$

The function $g(t, \tau)$ is the *impulse response* of the system. Over $t = \tau$ it will coincide with the weighting pattern considered in the previous section. However, note that unlike the weighting pattern, which is defined for all values of t and τ, the impulse response is defined only for $t \geq \tau$. For time-invariant systems $g(t, \tau) = g(t - \tau)$. For this reason, the function $\hat{g}(t,\tau)=g(t,t-\tau)$ is sometimes preferred.

For a general input $u(t)$ defined for $t \geq t_0$, the output is

$$y(t)=\int_{t_0}^{t}g(t,\tau)u(\tau)d\tau=\int_{t_0}^{t}\hat{g}(t,t-\tau)u(\tau)d\tau \tag{6}$$

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Biographical Sketch

Pablo A. Iglesias was born in Caracas, Venezuela in 1964. He received his B.A.Sc. degree in Engineering Science from the University of Toronto in 1987, and his Ph.D. in Control Engineering from Cambridge University in 1991. Since then, he has been on the faculty of the Department of Electrical and Computer Engineering at the Johns Hopkins University where he currently holds the title of Professor. He is also a member of the Center for Computational Medicine and Biology. He has had visiting appointments at Lund University, the Weizmann Institute of Science and the California Institute of Technology. He is also the co-author of the monograph *Minimum Entropy Control for Time-Varying Systems*. His current research interests include the control of time-varying systems, robust control theory and the use of control theory to study biological signaling.