ROBUSTNESS UNDER REAL PARAMETER UNCERTAINTY

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Contents

Summary

These collective results present the problem of analyzing and designing control systems containing uncertain parameters. These were developed since the mid 1980's following the publication of a breakthrough in 1978 on the stability of an interval polynomial family known as the Kharitonov Theorem. These fundamental results are aids in analyzing the behavior of control systems subject to real parameter uncertainty. The results specifically discussed here are the calculation of the real parametric stability margin, the Edge Theorem, Kharitonov's Theorem, the Generalized Kharitonov Theorem, and the determination of frequency domain templates for systems with uncertain parameters. These results are structural in nature and complement the H_2 and H_∞ approaches to control design based on optimality. An important characteristic of many of these results is that the theory picks out a small subset of points or lines in the parameter space where the "weakest" set of systems lie in terms of stability. With this set in hand, one can evaluate robustness of stability, worst-case stability margins and performances of the control system. These calculations, therefore, form an important

part of the control engineer's toolkit. Using these results, we also illustrate the application of these results to be designed by

a. showing how classical design techniques can be robustified, by using these results and
b. developing a new linear programming approach to controller design that exploits these results.

These examples should suggest to the reader how to formulate other design questions that take advantage of the fundamental results in Real Parametric Robust Control Theory.

1. Introduction

In the 1980's, a new approach to control systems began to emerge. This was based on several sharp results on the stability and performance of control systems subject to multiple uncertain real parameters. This theory complements the standard control theory based on the H_∞, H_2 l^1 and μ optimal control methods, which, by and large, did not deal with real parameter uncertainty. The results obtained in this area, which is called Real Parametric Robust Control Theory (RPRCT) here, reveal some new and interesting extremal properties of control systems that give insight into and aid in the control design process. The central results of this theory point out how the mathematics can be used to obtain useful information from the point of view of control systems.

We first show how the stability radius in the space of uncertain parameters may be calculated. The calculation is exact when these parameters appear linearly or affinely in the closed loop characteristic polynomial coefficients. When these coefficients appear multilinearly in the characteristic polynomial coefficients, the stability radius may be calculated to any degree of accuracy. Several applications of this calculation are demonstrated through examples.

Next, we present some extremal results which are very useful for analysis and design. These are respectively the Edge Theorem and the Generalized Kharitonov Theorem. These results allow us to evaluate the robust stability and performance of various systems subject to real parameter uncertainty in a computationally efficient manner. Then, how these results can aid in the construction of frequency domain templates is shown. Several control system applications of these results are also given where we show how classical design techniques can be robustified. This combines the advantages of classical and modern control.

2. Notations and Preliminaries

In this section, we introduce some basic notation and terminology that will be used throughout. The stability of linear time invariant control systems, is characterized by the root locations of the characteristic polynomial. Consider the standard feedback control system shown in Figure 1 consisting of a plant and controller connected in a feedback loop.

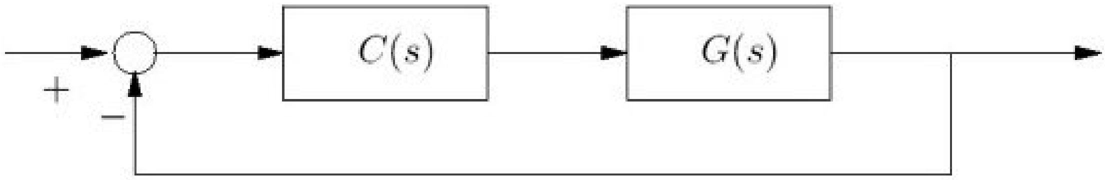

Figure 1: A standard feedback system

The plant and controller are assumed to be linear, time invariant dynamic systems with respective to real rational transfer function matrices $G(s)$ and $C(s)$. Let **p** denote a vector of physical plant parameters and **x** a vector of adjustable controller or design parameters. Write

$$C(s) = N_c(s,\mathbf{x})D_c^{-1}(s,\mathbf{x}) \tag{1}$$

$$G(s) = D_p^{-1}(s,\mathbf{p})N_p(s,\mathbf{p}) \tag{2}$$

where N_c, D_c, N_p and D_p are polynomial matrices in the complex variable s. Now the *characteristic polynomial* of the closed loop system is written as

$$\delta(s,\mathbf{x},\mathbf{p}) = \det\left[D_c(s,\mathbf{x})D_p(s,\mathbf{p}) + N_c(s,\mathbf{x})N_p(s,\mathbf{p})\right]. \tag{3}$$

Stability of the control system is equivalent to the condition that the roots of the characteristic polynomial all lie in a certain prescribed region S of the complex plane. For continuous time systems, the stability region S is the open left half, $\mathbb{C}^-$, of the complex plane and for discrete time systems; it is the open unit disc, that is a circle of radius unity, denoted $\mathbb{D}^1$, which is centered at the origin. In the control literature, stability of continuous time systems or left half plane stability is referred to as *Hurwitz stability*; and stability of discrete time systems or unit circle stability is referred to as *Schur stability*.

To clarify the above notation, a PID controller, for example, has a transfer function

$$C(s,\mathbf{x}) = K_p + \frac{K_I}{s} + K_D s \tag{4}$$

where the controller parameter vector is

$$\mathbf{x} = \begin{bmatrix} K_p & K_I & K_D \end{bmatrix} \tag{5}$$

Suppose the plant has transfer function $G(s)$ is written in two alternate parametrized

$$G(s,\mathbf{p_1}) = \frac{\mu(s-\alpha)}{(s-\beta)(s-\gamma)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0} = G(s,\mathbf{p_2}) \qquad (6)$$

with

$$\mathbf{p_1} = [\mu \quad \alpha \quad \beta \quad \gamma] \qquad (7)$$

and

$$\mathbf{p_2} = [a_0 \quad a_1 \quad b_0 \quad b_1 \quad b_2]. \qquad (8)$$

The characteristic polynomial is representable as

$$\delta(s,\mathbf{p_1},\mathbf{x}) = s(s-\beta)(s-\gamma) + \mu(s-\alpha)\left(K_p s + K_I + K_D s^2\right). \qquad (9)$$

or

$$\delta(s,\mathbf{p_2},\mathbf{x}) = s\left(b_2 s^2 + b_1 s + b_0\right) + (a_1 s + a_0)\left(K_p s + K_I + K_D s^2\right). \qquad (10)$$

2.3. Parametric Uncertainty

In general, mathematical models represent approximations to the real world and, therefore, it is appropriate to assume that the parameters appearing in such models actually lie in a range or interval of numerical values representing the uncertainty associated with that parameter. As linear models are supposed to account for the nonlinear behavior of the systems, these intervals may be large.

In the example, treated previously, the uncertainty in the plant model may be expressed in terms of uncertainty in the gain μ and the pole and zero locations α, β, γ(see Eq. (6)). Alternatively, it may be expressed in terms of the transfer function coefficients a_0, a_1, b_0, b_1, b_2. Each of these sets of plant parameters is subject to variation and may be assumed to lie in intervals.

In many control systems, the plant parameters may vary over a wide range about a nominal value $\mathbf{p}^0$. *Robust parametric stability* refers to the ability of a control system to maintain stability despite such large variations. During the design phase, the parameters $\mathbf{x}$ of a controller are regarded as adjustable variables and robust stability with respect to these parameters, also, is desirable in order to allow for adjustments to a nominal design to accommodate other design constraints.

If the controller is given, the *maximal* range of variation of the parameter $\mathbf{p}$, measured in a suitable norm, for which closed loop stability is preserved is the *parametric stability margin*. In other words

$$\rho_x := \sup\left\{\alpha : \delta\left(s,\mathbf{x},\mathbf{p}\right) \text{ stable}, \left\|\mathbf{p}-\mathbf{p}^0\right\| < \alpha\right\} \tag{11}$$

is the parametric stability margin of the system with the controller **x**. This is a quantitative measure of the performance of the controller **x**. Since ρ represents the *maximal perturbation*, it is indeed a legitimate quantitative measure by which one can compare the robustness of two proposed controller designs $\mathbf{x}_1$ and $\mathbf{x}_2$. This calculation is an important aid in analysis and design, much as gain and phase margin, or the value of a cost function or performance index is in optimal control. In the next subsection, we describe an important computational tool that can be used to evaluate ρ.

2.4. Boundary Crossing and Zero Exclusion

The fundamental notions of *boundary crossing* and *zero exclusion* play an important role in robust control. They depend on continuity of the roots of the polynomial on a parameter. For example, in the space of coefficients of a polynomial of degree n consider a path connecting a stable polynomial to an unstable one. Assuming the degree remains invariant on this path, that is, the number of roots is preserved, the first unstable polynomial encountered on this path must have some roots on the boundary of the stability region and the rest of the roots in the interior of the stability region. This result is called the *Boundary Crossing Theorem*. The computational version of this theorem is known as the *Zero Exclusion Condition* and is described as follows.

Consider the family of polynomials $\delta(s, \mathbf{p})$ of degree n, where the real parameter **p** ranges over a connected set Ω. Let the stability region in the complex plane be denoted as S with boundary ∂S. Suppose it is known that one member of the family is stable. Then, a useful technique of verifying robust stability of the family is to ascertain that no member of the family has a root on the stability boundary S. This can be done by checking that

$$\delta\left(s^*,\mathbf{p}\right) \neq 0, \quad \text{for all } \mathbf{p} \in \Omega, \quad s^* \in \partial S. \tag{12}$$

This can also be written as the *zero exclusion condition*

$$0 \notin \delta\left(s^*,\Omega\right), \quad \text{for all} \quad s^* \in \partial S. \tag{13}$$

The parametric stability margin may be computed by finding the smallest perturbation of $\mathbf{p}^0$ which results in a root just crossing the boundary, equivalently when the zero exclusion just begins to fail. The previous condition can be easily verified when the uncertainty set Ω is a box and the parameter **p** appears linearly or multilinearly in the characteristic polynomial coefficients. In the first case, the image set $\delta(s^*, \Omega)$ is itself a convex polygon and in the latter case it lies in the convex hull of the image of Ω. In these cases, the zero exclusion condition can be verified easily and so can stability margins. Motivated by such examples, the majority of robust parametric stability results are directed towards the *linear* and *multilinear* dependency cases, which fortunately, fit many practical applications.

The robust controller synthesis problem which is the problem of determining **x** to achieve stability and a prescribed level of parametric stability margin ρ is unfortunately as yet unsolved. In an engineering sense, however, many effective techniques exist for robust parametric controller design. In particular, the exact calculation of ρ_x can itself be used in an iterative loop to adjust **x** to robustify the system. In the next section, we derive the procedure to compute ρ in some detail.

3. Real Parameter Stability Margin

In this section, we show how the parametric stability margin can be computed in the case in which the characteristic polynomial coefficients depend affinely on the uncertain parameters. In such cases, we may write

$$\delta(s,\mathbf{p}) = a_1(s)\,p_1 + \cdots + a_l(s)\,p_l + b(s) \tag{14}$$

where $a_i(s)$ and $b(s)$ are real polynomials and the parameters p_i are real. Write **p** for the vector of uncertain parameters, $\mathbf{p}^0$ the nominal parameter vector and $\Delta\mathbf{p}$ the perturbation vector. In other words

$$\mathbf{p} = \begin{bmatrix} p_1 & p_2 & \cdots & p_l \end{bmatrix} \tag{15}$$

$$\mathbf{p}^0 = \begin{bmatrix} p_1^0 & p_2^0 & \cdots & p_l^0 \end{bmatrix} \tag{16}$$

$$\Delta\mathbf{p} = \begin{bmatrix} p_1 - p_1^0 & p_2 - p_2^0 & \cdots & p_l - p_l^0 \end{bmatrix} = \begin{bmatrix} \Delta p_1 & \Delta p_2 & \cdots & \Delta p_l \end{bmatrix}. \tag{17}$$

the characteristic polynomial can be written as

$$\delta(s,\mathbf{p}^0 + \Delta\mathbf{p}) = \underbrace{\delta(s,\mathbf{p}^0)}_{\delta^0(s)} + \underbrace{a_1(s)\Delta p_1 + \cdots + a_l(s)\Delta p_l}_{\Delta\delta(s,\Delta\mathbf{p})}. \tag{18}$$

Let s^* denote a point on the stability boundary ∂S. For $s^* \in \partial S$ to be a root of $\delta(s,\mathbf{p}^0 + \Delta\mathbf{p})$ we must have

$$\delta(s^*,\mathbf{p}^0) + a_1(s^*)\Delta p_1 + \cdots + a_l(s^*)\Delta p_l = 0. \tag{19}$$

In many instances, it is important to consider weighted perturbations, to account for, say, scaling factors, units, or normalization. Letting $w_i > 0$, $i = 1, \cdots, l$ denote a set of weights rewrite the above equation as follows:'

$$\delta(s^*,\mathbf{p}^0) + \frac{a_1(s^*)}{w_1} w_1\Delta p_1 + \cdots + \frac{a_l(s^*)}{w_l} w_l\Delta p_l = 0. \tag{20}$$

The minimum norm solution of this equation gives us $\rho(s^*)$:

$$\rho\left(s^*\right) = \inf\left\{ \|\Delta\mathbf{p}\|^w : \delta\left(s^*, \mathbf{p}^0\right) + \frac{a_1\left(s^*\right)}{w_1} w_1 \Delta p_1 + \cdots + \frac{a_l\left(s^*\right)}{w_l} w_l \Delta p_l = 0 \right\}. \tag{21}$$

The equation corresponding to loss of degree is:

$$\delta_n\left(\mathbf{p}^0 + \Delta\mathbf{p}\right) = 0. \tag{22}$$

If a_{in} denotes the coefficient of the n^{th} degree term in the polynomial $a_i(s)$, $i = 1, 2, \cdots, l$ the above equation becomes

$$\underbrace{a_{1n} p_1^0 + a_{2n} p_2^0 + \cdots + a_{ln} p_l^0}_{\delta_n\left(\mathbf{p}^0\right)} + a_{1n}\Delta p_1 + a_{2n}\Delta p_2 + \cdots a_{ln}\Delta p_l = 0. \tag{23}$$

or, after introducing the weight $w_i > 0$

$$\underbrace{a_{1n} p_1^0 + a_{2n} p_2^0 + \cdots + a_{ln} p_l^0}_{\delta_n\left(\mathbf{p}^0\right)} + \frac{a_{1n}}{w_1} w_1 \Delta p_1 + \frac{a_{2n}}{w_2} w_2 \Delta p_2 + \cdots + \frac{a_{ln}}{w_l} w_l \Delta p_l = 0 \tag{24}$$

The minimum norm $\|\Delta\mathbf{p}\|^w$ solution of this equation gives us ρ_d.
The Eq. (24) is real and can be rewritten in the form

$$\underbrace{\begin{bmatrix} \frac{a_{1n}}{w_1} & \cdots & \frac{a_{ln}}{w_l} \end{bmatrix}}_{A_n} \underbrace{\begin{bmatrix} w_1 \Delta p_1 \\ \vdots \\ w_l \Delta p_l \end{bmatrix}}_{t_n} = \underbrace{-\delta_n^0}_{b_n}. \tag{25}$$

In Eq. (20), two cases may occur depending on whether s^* is real or complex. If $s^* = s_r$ where s_r is real, we have the single equation

$$\underbrace{\begin{bmatrix} \frac{a_1\left(s_r\right)}{w_1} & \cdots & \frac{a_l\left(s_r\right)}{w_l} \end{bmatrix}}_{A(s_r)} \underbrace{\begin{bmatrix} w_1 \Delta p_1 \\ \vdots \\ w_l \Delta p_l \end{bmatrix}}_{t(s_r)} = \underbrace{-\delta^0\left(s_r\right)}_{b(s_r)}. \tag{26}$$

Let x_r and x_i denote the real and imaginary parts of a complex number x, i.e.,

$$x = x_r + jx_i \quad \text{with } x_r, x_i \text{ real}. \tag{27}$$

so that

$$a_k\left(s^*\right) = a_{kr}\left(s^*\right) + ja_{ki}\left(s^*\right) \tag{28}$$

and

$$\delta^0\left(s^*\right) = \delta_r^0\left(s^*\right) + j\delta_i^0\left(s^*\right). \tag{29}$$

If $s^* = s_c$, where s_c is complex, Eq. (20) is equivalent to two equations which can be written as follows:

$$\underbrace{\begin{bmatrix} \dfrac{a_{1r}\left(s_c\right)}{w_1} & \cdots & \dfrac{a_{lr}\left(s_c\right)}{w_l} \\ \dfrac{a_{1i}\left(s_c\right)}{w_1} & \cdots & \dfrac{a_{li}\left(s_c\right)}{w_l} \end{bmatrix}}_{A(s_c)} \underbrace{\begin{bmatrix} w_1\Delta p_1 \\ \vdots \\ w_l\Delta p_l \end{bmatrix}}_{t(s_c)} = \underbrace{\begin{bmatrix} -\delta_r^0\left(s_c\right) \\ -\delta_i^0\left(s_c\right) \end{bmatrix}}_{b(s_c)}. \tag{30}$$

These equations completely determine the parametric stability margin in any norm. Let $t^*(s_c)$, $t^*(s_r)$, and t_n^* denote the minimum norm solutions of Eqs. (30), (26), and (25), respectively. Thus,

$$\left\|t^*\left(s_c\right)\right\| = \rho\left(s_c\right) \tag{31}$$

$$\left\|t^*\left(s_r\right)\right\| = \rho\left(s_r\right) \tag{32}$$

$$\left\|t_n^*\right\| = \rho_d. \tag{33}$$

If any of the above Eqs. (25) - (30) do not have a solution, the corresponding value of $\rho(\cdot)$ is set equal to infinity.

Let ∂S_r and ∂S_c denote the real and complex subsets of ∂S:

$$\partial S = \partial S_r \cup \partial S_c. \tag{34}$$

$$\rho_r := \inf_{s_r \in \partial S_r} \rho\left(s_r\right) \tag{35}$$

$$\rho_c := \inf_{s_c \in \partial S_c} \rho\left(s_c\right). \tag{36}$$

Finally, the real parametric stability margin is:

$$\rho = \inf\left\{\rho_r, \rho_c, \rho_d\right\}. \tag{37}$$

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Biographical Sketch

L. H. Keel was born in Pusan, Korea in 1953. He received his B.S. degree in electronic engineering from Korea University, Seoul in 1978, and his M.S. and Ph.D. degrees in electrical engineering from Texas A&M University, College Station, Texas in 1983 and 1986, respectively. Since then he has been with the Center of Excellence in Information Systems at Tennessee State University, Nashville, Tennessee, where he is now Professor of Electrical Engineering and Director of the Center for System Science Research. His research interests include robust control, system identification, structure and control, and computer-aided control system design. He has authored and co-authored numerous technical papers in the field of control systems and two books, including *Robust Control: The Parametric Approach*, Upper Saddle River, NJ: Prentice-Hall, 1995.