CONTROLLABILITY AND OBSERVABILITY OF DISTRIBUTED PARAMETER SYSTEMS

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Contents

Summary

Notions of controllability and observability are of great importance in modern mathematical control theory. It should be stressed that the theory of controllability and observability of linear control systems is relatively well developed. In this article various concepts of controllability and observability of linear, infinite-dimensional, continuous-time control systems with constant coefficients are studied. The article contains various fundamental definitions, theorems, and corollaries concerning approximate controllability, approximate boundary controllability, and observability. Moreover, the different relations among them are also listed. As illustrations of the general theory, constrained approximate controllability, constrained approximate boundary controllability, and observability of linear distributed parameter dynamic systems described by partial differential state equations and linear output equation are considered.

1. Introduction

In recent decades modern control theory of linear dynamic systems has been the subject of considerable interest by many research scientists. It has been motivated, on the one hand, by the wide range of applications of linear models in various areas of science and engineering, and on the other hand, by the difficult and stimulating theoretical problems posed by such systems. The article is intended to provide information about various

fundamental research results obtained in the fields of controllability and observability of linear distributed parameter systems, and discusses their significance and consequences.
At the end of the 1960s, the state space theory of linear control systems for both time-invariant and time-varying dynamic systems had essentially been worked out. The basic concepts of controllability and observability, and the weaker notions of stabilizability and detectability, play an essential, fundamental role in the solutions of many important different optimal control problems. Controllability and observability are two of the fundamental concepts in modern mathematical control theory. Many dynamic systems are such that the control does not affect the complete state of the dynamic system but only a part of it. On the other hand, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamic system. Therefore, it is very important to determine whether or not control and observation of the complete state of the dynamic system are possible. Roughly speaking, controllability generally means that it is possible to steer a dynamic system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. On the other hand, observability means that it is possible to recover uniquely the initial state of the dynamic system from a knowledge of the input and output. Controllability and observability play an essential role in the development of modern mathematical control theory. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability.

In the literature there are many different definitions of controllability and observability that depend on the type of dynamic system. A growing interest has been developed over the past few years in problems involving signals and systems that are defined in infinite-dimensional linear spaces. The most popular examples of infinite-dimensional dynamic systems are distributed parameter systems and dynamic systems with delays. The motivations for studying distributed parameter systems have been well justified in several papers and monographs. Most of the major results concerning infinite dimensional dynamic systems are developed for the linear case. During the last two decades controllability and observability of distributed parameter systems have been considered in many papers and books. The main purpose of this article is to present a compact review of the existing controllability and observability results, mainly for linear distributed parameter dynamic systems. The majority of the results in this area concerns linear systems with constant coefficients. It should be pointed out that for linear systems, controllability and observability conditions sometimes have pure algebraic forms and are rather easily computable. These conditions require verification of the rank conditions for suitable defined constant controllability and observability matrices.

2. Controllability of Infinite-Dimensional Systems

2.1. Mathematical Model

In this section we study the linear abstract control system:

$$x'(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \text{ for } t\in[0,\mathrm{T}] \tag{1}$$

$$\text{with initial condition } x(0)\in \mathrm{X} \tag{2}$$

where the state $x(t)$ takes values in a real infinite-dimensional separable Hilbert space X and the values of the control $u(t)$ are in the space $U = R^m$.

Let us assume that the linear, generally unbounded, operator A generates a strongly differentiable semigroup $S(t)$ on X for $t \geq 0$, and B is a linear bounded operator from the space R^m into X. Therefore, operator $B = [b_1, b_2, \ldots, b_j, \ldots, b_m]$ and:

$$Bu(t) = \sum_{j=1}^{j=m} b_j u_j(t)$$

where $b_j \in X$ for $j = 1,2, \ldots, m$, and $u(t) = [u_1(t), u_2(t), \ldots, u_j(t), \ldots, u_m(t)]^{tr}$

We would like to emphasize that the assumption that linear operator B is bounded rules out the application of our theory to boundary control problems, because in this situation B is typically an unbounded operator.

Let $U_c \subset U$ be a closed convex cone with a nonempty interior and vertex at zero. The set of admissible controls for the dynamic system (1) is $U_{ad} = L_\infty([0,T], U_c)$.

Then for a given admissible control u(t) there exists a unique so-called mild solution x(t;x(0),u) of the equation (1) with initial condition (2) described by the integral formula:

$$x(t;u) = \int_0^t S(t-s)(F(x(s;u)) + Bu(s))ds$$

2.2. Controllability Conditions

For the linear abstract dynamic system (1), it is possible to define many different concepts of controllability. In the sequel we shall focus our attention on the so-called constrained exact controllability in the time interval [0,T]. In order to do that, first of all let us introduce the notion of the attainable set at time T > 0 from the zero initial state $x(0) = 0$, denoted by $K_T(U_c)$ and defined as follows:

$K_T(U_c) = \{x \in X : x = x(T,0,u) ,\ u(t) \in U_c \text{ for a.e. } t \in [0,T]\}$

where $x(t,0,u)$, $t > 0$ is the unique solution of Eq. (1) with zero initial condition and control u. Moreover, let us denote:

$$K_\infty(U_c) = \bigcup_{t>0} K_t(U_c)$$

Now, using the above concepts of the attainable sets, let us recall the familiar definitions of constrained exact controllability for dynamic system (1).

Definition 1: The dynamic system (1) is said to be U_c-approximately controllable in

[0,T] if the attainable set $K_T(U_c)$ is dense in the space X.

Definition 2: The dynamic system (1) is said to be U_c-approximately controllable if the attainable set $K_\infty(U_c)$ is dense in the space X.

Definition 3: The dynamic system (1) is said to be U_c-exactly controllable in [0,T] if $K_T(U_c) = X$.

Definition 4: The dynamic system (1) is said to be U_c-exactly controllable if $K_\infty(U_c) = X$.

Approximate controllability in [0,T] implies approximate controllability, and similarly, exact controllability in [0,T] implies exact controllability. Moreover, exact controllability always implies approximate controllability. However, conditions for exact controllability are rather restrictive, therefore, in the sequel only approximate controllability will be considered. Let us observe, that for the finite-dimensional case (i.e., when the state space $X = R^{n)}$, we may omit the words "approximate" and "exact" in the above definitions, since in this case exact controllability is equivalent to approximate controllability.In order to obtain computable criteria for approximate controllability we shall concentrate on dynamic systems defined in a separable infinite-dimensional Hilbert space X with a normal, generally unbounded, operator A with a compact resolvent. Hence, the operator A has only a pure discrete-point spectrum consisting of an infinite sequence $\{s_i\}$, $i = 1,2,3, \dots$ of distinct isolated eigenvalues of A, each with finite multiplicity r_i. Moreover, in the space X there is a corresponding complete orthonormal set $\{x_{ij}\}$ $i = 1,2,3, \dots, j = 1,2,3, \dots r_i$ of eigenvectors of the operator A. Therefore, the semigroup S(t) is given by:

$$S(t)x = \sum_{i=1}^{i=\infty} \exp(s_i t) \sum_{j=1}^{j=r_i} \left\langle x, x_{ij} \right\rangle_X x_{ij} \quad , \quad for \ t \geq 0 \ and \ x \in X$$

The class of operators satisfying the above assumptions arises in classical control problems for linear distributed parameter systems. In order to formulate computable, constrained, approximate controllability conditions let us denote B_i, $i = 1,2,3, \dots$ $(r_i \times m)$-dimensional constant matrices:

$$B_{qi} = \begin{bmatrix} \left\langle b_{q1}, x_{i1} \right\rangle_X & \left\langle b_{q2}, x_{i1} \right\rangle_X & \dots & \left\langle b_{qp}, x_{i1} \right\rangle_X & \dots & \left\langle b_{qm}, x_{i1} \right\rangle_X \\ \left\langle b_{q1}, x_{i2} \right\rangle_X & \left\langle b_{q2}, x_{i2} \right\rangle_X & \dots & \left\langle b_{qp}, x_{i2} \right\rangle_X & \dots & \left\langle b_{qm}, x_{i2} \right\rangle_X \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \left\langle b_{q1}, x_{ij} \right\rangle_X & \left\langle b_{q2}, x_{ij} \right\rangle_X & \dots & \left\langle b_{qp}, x_{ij} \right\rangle_X & \dots & \left\langle b_{qm}, x_{ij} \right\rangle_X \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \left\langle b_{q1}, x_{ir_i} \right\rangle_X & \left\langle b_{q2}, x_{ir_i} \right\rangle_X & \dots & \left\langle b_{qp}, x_{ir_i} \right\rangle_X & \dots & \left\langle b_{qm}, x_{ir_i} \right\rangle_X \end{bmatrix} \tag{3}$$

For the case when all the eigenvalues s_i are simple (i.e., $r_i = 1$), for $i = 1,2,3, \ldots$, B_i are m-dimensional row vectors of the following form:

$$B_i = \left[\langle b_1, x_i \rangle_X , \langle b_2, x_i \rangle_X ,..., \langle b_j, x_i \rangle_X ,..., \langle b_m, x_i \rangle_X \right] \quad \text{for } i = 1,2,3,... \tag{4}$$

Theorem 1: Let X be a separable Hilbert space and assume that the operator A is normal with compact resolvent. Let U = R^m and U_c be a cone in R^m with vertex at the origin such that int(coU_c) ≠ ∅. Then the linear dynamic system (1) and (2) is approximately U_c-controllable in finite time if and only if the following conditions hold:

$$B_{Ti} coU_c = R^{r_i} \quad , \quad for\ all\ i = 1,2,3,...$$

whenever s_i is a real eigenvalue,

rank$B_i = r_i$, for $i = 1,2,3, \ldots$

whenever s_i is a complex eigenvalue.

Corollary 1: Let U = U_c = R^m. Then the linear dynamic system (1) is approximately R^m-controllable in finite time if and only if the following conditions hold:
rank$B_i = r_i$, for $i = 1,2,3, \ldots$

Corollary 2: Let U = U_c = R^m and $r_i = 1$ for $i = 1,2,3, \ldots$. Then the linear dynamic system (in Section 2.1) is approximately R^m-controllable in finite time if and only if the following conditions hold:

$$\sum_{j=1}^{j=m} \langle b_j, x_i \rangle^2_{L^2(D)} \neq 0 \text{ for } i = 1,2,3, \ldots$$

or equivalently $b^i \neq 0$, for $i = 1,2,3, \ldots$

It should be pointed out that the multiplicities r_i, $i = 1,2,3, \ldots$ of the eigenvalues s_i , $i = 1,2,3, \ldots$ are finite for every i; however, we do not always have sup $r_i < \infty$. The number $r = \sup r_i < \infty$ has an important meaning in the investigation of approximate controllability.

Corollary 3: If $r = \sup r_i = \infty$, then the dynamic system (1) is not R^m-approximately controllable.

The quite general controllability conditions given in Theorem 1 and Corollaries 1 and 2, can be used to formulate controllability criteria for distributed parameter dynamic systems described by the linear partial differential state equations.

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Biographical Sketch

Jerzy Klamka was born in Poland in 1944. He received M.S. and Ph.D. degrees in Control Engineering from the Silesian Technical University in Gliwice, Poland, in 1968 and 1974 respectively. He also received M.S. and Ph.D. degrees in Mathematics from the Silesian University in Katowice, Poland, in 1971 and 1978 respectively. In 1981 he received habilitation in Control Engineering, and in 1990 became titular professor in Control Engineering at the Silesian Technical University in Gliwice, Poland.

Since 1968 he has been working for the Institute of Control Engineering of the Silesian Technical University in Gliwice, where he is now a full professor. In 1973 and 1980 he taught semester courses in mathematical control theory at the Stefan Banach International Mathematical Center in Warsaw. He has been a member of the American Mathematical Society (AMS) since 1976, and the Polish Mathematical Society (PTM) since 1982. He is also a permanent reviewer for *Mathematical Reviews* (since 1976) and for *Zentralblatt für Mathematik* (since 1982). In 1981 and 1991 he was awarded Polish Academy of Sciences awards. In 1978, 1982, and 1990 he received awards from the Ministry of Education. While in 1994 he was awarded the Polish Mathematical Society award.

In 1991 he published the monograph *Controllability of Dynamical Systems*. In the last 30 years he has published more than 100 papers in international journals, for example, in *IEEE Transactions on Automatic Control*; *Automatica*; *International Journal of Control*; *Journal of Mathematical Analysis and Applications*; *Systems and Control Letters*; *Foundations of Control Engineering*; *Systems Science*; *Kybernetika*; *IMA Journal on Mathematical Control and Information*; *Nonlinear Analysis*; *Theory, Methods and Applications*; *Systems Analysis*; *Modelling*; *Simulation*; *Archives of Control Science*; *Applied Mathematics and Computer Science*; *Advances in Systems Science and Applications*; *Bulletin of the Polish Academy of Sciences*; *Mathematical Population Dynamics*; *Lecture Notes in Control and Information Sciences*; *Analele Universitati din Timisoara*; and *Acta Mathematicae Silesiane*. He has taken part in many international congresses, conferences, and symposiums. His major current interest is controllability theory for linear and nonlinear dynamic systems, and in particular controllability of distributed parameter systems, dynamic systems with delays, and multidimensional discrete systems.