FEEDBACK LINEARIZATION OF NONLINEAR SYSTEMS

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Contents

Summary

The chapter examines the feedback linearization problem. Relying on the concept of relative degree, a change of coordinates and a feedback law are found for which the closed-loop system in the new coordinates is in a normal form. This is the point of departure for obtaining a constructive procedure allowing us to transform a nonlinear system into a linear and controllable one.

1. The Problem of Feedback Linearization

Consider a single-input single-output nonlinear system modeled by differential equations of the type

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measured output and $f(x), g(x), h(x)$ are smooth functions of x.

A basic issue in control theory is how to use feedback in order to modify the original internal dynamics of a controlled plant in such a way as to obtain the same behavior of some prescribed *autonomous linear system*.

This problem, which in the case of linear systems is known as the problem of pole placement, is known in the more general framework of nonlinear systems as *feedback linearization* (see Bibliography).

Changes in the description and in the behavior of system (1) will be considered under two types of transformation: (i) changes of coordinates in the state space and (ii) static state feedback control laws, i.e. *memoryless* state feedback laws.

In the case of a *linear* system,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2}$$

a static state feedback control law takes the form

$$u = Fx + Gv\,, \tag{3}$$

in which v represents a new control input and F and G are matrices of appropriate dimensions. Moreover, only linear changes of coordinates are usually considered. This corresponds to the substitution of the original state vector x with a new vector z related to x by a transformation of the form

$$z = Tx\,,$$

where T is a nonsingular matrix. Accordingly, the original description of system (2) is replaced by a new description

$$\begin{aligned} \dot{z} &= \tilde{A}z + \tilde{B}u \\ y &= \tilde{C}z \end{aligned} \tag{4}$$

in which

$$\tilde{A} = TAT^{-1}\,, \quad \tilde{B} = TB\,, \quad \tilde{C} = CT^{-1}\,.$$

In the case of a nonlinear system, a static feedback control law is a control law of the form

$$u = \alpha(x) + \beta(x)v\,, \tag{5}$$

where v represents a new control input and $\beta(x)$ is assumed to be nonzero for all x. Moreover, *nonlinear* changes of coordinates are considered, i.e., transformations of the form

$$z = \Phi(x)\,, \tag{6}$$

where z is the new state vector and $\Phi(x)$ represents a (n-vector valued) function of n variables,

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \cdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} \phi_1(x_1, x_2, \ldots, x_n) \\ \phi_2(x_1, x_2, \ldots, x_n) \\ \cdots \\ \phi_n(x_1, x_2, \ldots, x_n) \end{pmatrix},$$

with the following properties:

1. $\Phi(x)$ is invertible; i.e. there exists a function $\Phi^{-1}(z)$ such that
   $\Phi^{-1}(\Phi(x)) = x$, $\Phi(\Phi^{-1}(z)) = z$
   for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{R}^n$.
2. $\Phi(x)$ and $\Phi^{-1}(z)$ are both smooth mappings.

A transformation of this type is called a *global diffeomorphism*. The first property is needed to guarantee the invertibility of the transformation to yield the original state vector as

$$x = \Phi^{-1}(z),$$

while the second one guarantees that the description of the system in the new coordinates is still a smooth one.

Sometimes a transformation possessing both of these properties and defined for all x is hard to find and the properties in question are difficult to check. Thus, in most cases, transformations defined only in the neighborhood of a given point are of interest.

A transformation of this type is called a *local diffeomorphism*. To check whether or not a given transformation is a local diffeomorphism, the following result is very useful.

Proposition 1. *Suppose* $\Phi(x)$ *is a smooth function defined on some open subset* $U \in \mathbb{R}^n$. *Suppose the Jacobian matrix*

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{pmatrix}$$

is nonsingular at point $x = x^0$. *Then, for some suitable open subset* U^0 *of* U, *containing* x^0, $\Phi(x)$ *defines a local diffeomorphism between* U^0 *and its image*

$\Phi(U^0)$.

The effect of a change of coordinates on the description of a nonlinear system can be analyzed as follows. Set

$$z(t) = \Phi(x(t))$$

and differentiate both sides with respect to time to yield

$$\dot{z}(t) = \frac{dz}{dt} = \frac{\partial \Phi}{\partial x}\frac{dx}{dt} = \frac{\partial \Phi}{\partial x}(f(x(t)) + g(x(t))u(t)).$$

Then, expressing $x(t)$ as $\Phi^{-1}(z(t))$, one obtains

$$\begin{aligned} \dot{z}(t) &= \tilde{f}(z(t)) + \tilde{g}(z(t))u(t) \\ y(t) &= \tilde{h}(z(t)), \end{aligned}$$

where

$$\tilde{f}(z) = \left(\frac{\partial \Phi}{\partial x} f(x)\right)_{x=\Phi^{-1}(z)}, \quad \tilde{g}(z) = \left(\frac{\partial \Phi}{\partial x} g(x)\right)_{x=\Phi^{-1}(z)},$$
$$\tilde{h}(z) = \left(h(x)\right)_{x=\Phi^{-1}(z)}.$$

The latter are the formulas relating the new description of the system to the original one.

Given the nonlinear system (1), the problem of *feedback linearization* consists of finding, if possible, a change of coordinates of the form (6) and a static state feedback of the form (5) such that the composed dynamics of (1) and (5), namely the system

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v, \qquad (7)$$

expressed in the new coordinates z, is the linear and controllable system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v. \end{aligned}$$

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Biographical Sketches

Alberto Isidori was born in Rapallo, Italy, in 1942. His research interests are primarily focused on mathematical control theory and control engineering. He graduated in electrical engineering from the University of Rome in 1965. Since 1975, he has been Professor of Automatic Control in this University. Since 1989, he also holds a part-time position of Professor of Systems Science and Math. at Washington University in St. Louis. He has held visiting positions at various academic/research institutions which include: University of Florida, Gainesville (November 1974), Washington University, St. Louis (August-October 1980, August-December 1983), University of California, Davis (July-August 1983), Arizona State University, Tempe (August-December 1986, April-May 1989), University of Illinois, Urbana (April-May 1987), CINVESTAV, Mexico City (September 1987), University of California, Berkeley (January 1988), CNRS, Paris (May 1988), ETH, Zurich (April-May 1991), Universite Paris-Dauphine, Paris (May 1992), NASA, Langley (November 1996, February 1997).

He is the author of several books: Teoria dei Sistemi (in Italian), with A.Ruberti, 1979; Sistemi di Controllo (in Italian), 1979 and 1992; Nonlinear Control Systems (Springer Verlag), 1985, 1989 and 1995; Topics in Control Theory (Birkhauser), with H.Knobloch and D.Flockerzi, 1993; Output Regulation of Uncertain Nonlinear Systems (Birkhauser), with C.I. Byrnes and F. Delli Priscoli, 1997. He is also editor/coeditor of nine volumes of Conference proceedings and author of over 130 articles, for a large part on the subject of nonlinear feedback design.

He received the G.S.Axelby Outstanding Paper Award from the Control Systems Society of IEEE in 1981, for his technical contributions to the application of differential geometry to the problem of noninteracting control of nonlinear systems, and in 1990, for his technical contributions to the solution of the problem of asymptotic regulation and tracking in nonlinear systems. He also received from the IFAC the Automatica Prize in 1991 for his technical contributions to the application the notion of zero dynamics in problems of feedback stabilization. In 1987 he was elected Fellow member of the IEEE "for fundamental contributions to nonlinear control theory".

In 1996, at the opening of 13th IFAC World Congress in San Francisco, Dr. Isidori received the "Giorgio Quazza Medal". This medal is the highest technical award given by the International Federation of Automatic Control, and is presented once every third year for lifetime contributions to automatic control science and engineering. The "Giorgio Quazza Medal" was awarded to Dr. Isidori for "pioneering and fundamental contributions to the theory of nonlinear feedback control".

He has organized or co-organized several international Conferences on the subject feedback design for nonlinear systems. In particular, he was the initiator a permanent series of IFAC Symposia on this topic.

He is presently serving in numerous Editorial Boards of major archival journals, which include Automatica, IEEE Transactions on Automatic Control, International Journal of Control, Journal of Mathematical Systems Estimation and Control, International Journal of Robust and Nonlinear Control. He has also served in the program committee of several major international Conferences.

He acted as Program director, in the area of Systems and Control, for the Italian Department of Education from 1983 to 1989. From 1993 to 1996 he served in the Council of IFAC.

Claudio De Persis received his Laurea degree summa cum laude in Electrical Engineering and his doctoral degree in Systems Engineering in 1996 and, respectively, 2000 both from Università di Roma "La Sapienza", Rome, Italy. He held visiting positions in the Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX, and in the Department of Mathematics, University of California, Davis, CA in 1998-1999. From November 1999 to June 2001 he has been a Research Associate in the Department of Systems Science and Mathematics, Washington University in St. Louis, MO. Since July 2001 he has been a Postdoctoral Research Associate in the Department of Electrical Engineering, Yale University, New Haven, CT. On November 1, 2002, he took up his new position as Assistant Professor in the Department of Computer and Systems Science "A. Ruberti", Università di Roma "La Sapienza". He has given contributions to the theory of fault detection for nonlinear systems, switched systems and supervisory control with constraints. His current research interests include observation and control with limited information, hybrid systems, monitoring in large-scale systems, complex systems, networks, modern communication, post-genomic biology.