MODAL LOGIC AND ITS APPLICATIONS

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Summary

Modal logic is a broad and rapidly expanding area of logic with applications to such diverse areas as computer science, linguistics and philosophy. It deals with the logical behavior of such modal locutions as 'must' and 'might', 'was' and 'will', 'ought', and 'may'. It specifies formal languages within which such locutions may be encoded, it lays down axioms and rules by which the locutions are governed, it sets up an interpretation for the resulting symbolism, and it proves various general results concerning the system and its interpretation.

1. Introduction

Modal logic is the resulting logic of possibility and necessity and of other such notions. It began, as with logic in general, with Aristotle, who make some remarks on the 'modal syllogism'; and various notions and principles of modal logic were extensively discussed in the middle ages. But the subject came into its own only at the beginning of the twentieth century.

The American philosopher-logician, C. I. Lewis, was bothered by Russell's reading of the classical horseshoe '$\supset$' as 'implication' and by his saying such things as that a falsehood implied every proposition or that every proposition implied a truth and Lewis therefore proposed a 'strict' or modal reading of implication, now signified by '$\rightarrow$', in place of Russell's understanding of implication as 'material' or truth-functional. Lewis and Langford [8] proposed various axiomatic systems of modal logic of increasing strength - ranging from the weakest, S1, through S2, S3 and S4, to the strongest, S5. These systems provided the basis for much subsequent investigation, although the necessity- operator '$\Box$' soon supplanted other modal operators as the favored modal primitive ($\Box A$ might be defined as $\top \rightarrow A$, where $\top$ is a standard tautology, and, conversely, $A \rightarrow B$ might be defined as $\Box(A \supset B)$).

This work was almost entirely formal; it was largely concerned with the deductive development of the various formal systems. The next major breakthrough in the subject came with the development of a semantics. Classical logic has a natural 'modeling' or semantics, first made explicit by Tarski [13]. An interpretation, or model, specifies a domain of objects for the quantifiers to range over and an extension for each predicate of the language. Relative to a model, it can be precisely stated when a closed formula of the language is true. We may then define a formula to be *valid* when it is true under all models; and, given this conception of validity, it may then be shown that the valid formulas coincide with the theorems of the standard systems of classical predicate logic (the so-called 'completeness theorem'). (see Section 5.7 in *Formal Logic*)

However, it was not at all clear how to achieve anything comparable in the case of modal logic. The principal difficulty has to do with the original motivation for modal logic. Lewis was dissatisfied with Russell's truth-functional reading of implication; he did not think that the truth-value of '$S \rightarrow T$' depended solely upon the truth-values of S and T; and similarly the truth-value of '$\Box S$' does not solely depend upon the truth-value of S. But the truth-functional reading provides us with a natural basis for a semantics, since it enables us to determine the truth-values of complex sentential formulas on the basis of their components; and, in the absence of a truth-functional determination of truth-value, it is not clear what to put in its place.

Two main solutions to this problem were proposed. According to the first, of Tarski and his school [14], the two truth-values of classical logic - the True and the False - were to be replaced by a range of different truth-values. One might intuitively think of these values as the *propositions* expressed by the various formulas of the language though, from the formal point of view, they were simply regarded as elements from an arbitrary Boolean algebra. Given the assignment of the 'proposition' p to A, the proposition assigned to $\Box A$ could simply be taken to be the result $N(p)$ of applying an appropriate necessity-operation N to the proposition p . The various axioms of different modal logics would then correspond to various constraints on the operation. The axiom $\Box A \supset A$', for example, would correspond to the condition that $N(p) \leq p$, for $\leq$ the relation of inclusion within the Boolean algebra. Under a suitable choice of constraints, completeness for many of the standard systems of modal logic could then be established.

According to the other, later approach of Kripke and others [7], the two truth-values of classical logic were to be relativized to an index or 'world'. Thus instead of talking of the truth or falsehood of a formula simpliciter, we should talk of their truth-value *at* a world. Truth-functional formulas could then be evaluated in the usual manner at a world; $\sim A$, for example, could be taken to be true at a world just in case A was false at that world. However, in evaluating a modal formula at a world, we would have to take account of the truth-values of its components not only at that world but also at other worlds. In the simplest case, we could take $\Box A$, for example, to be true at a world just in case A was true at all worlds. But we might also want the worlds we look at in evaluating the formula to depend upon the world in question; $\Box A$ would then be taken to be true at a world just in case A was true at all suitably related (or 'accessible') worlds. Different axioms could then be seen to correspond to different constraints on the accessibility relation. The axiom $\Box A \supset A$, for example, would correspond to the accessibility relation being reflexive. Under a suitable choice of constraints on the accessibility relation, completeness for many of the standard systems of modal logic could again be established.

We might think of the two approaches to the semantics for modal logic as corresponding to two strategies for extending the semantics for classical logic. On the one hand, we might extend the semantics 'outwards', replacing the two truth-values with many alternative truth-values. On the other hand, we might extend the semantics 'downwards', making the assignment of truth-values relative to an index or point of evaluation. The first strategy was very natural at the time, since many-valued approaches to sentential logic had already been considered [12]. The second strategy was less natural within the context of sentential logic although it is, in a sense, already implicit in Tarski's original treatment of quantificational logic. For the assignment of a truth-value to an open formula (one containing free variables) must be taken to be relative to an assignment of values to those variables; and the evaluation of a quantified formula, such as $\forall xA(x)$, relative to an assignment must be seen to depend upon the evaluation of the component formula, $A(x)$, under suitably related assignments (those differing only in what is assigned to x). Thus the general idea of relativized evaluation is already implicit in the semantics for quantificational logic, though not in its application to the purely sentential case.

Since this breakthrough in the semantics, modal logic has developed in two different though complementary directions. On the one hand, workers in the field have developed and investigated a great variety of different systems with a great variety of different applications - to philosophy and linguistics, for example, as well as to computer science and artificial intelligence. On the other hand, they have conducted a general enquiry into the nature of these different systems. The emphasis here is not on this or that system but on the 'space' of systems as whole.

In what follows, I shall spell out the semantics to modal logic, using the possible-worlds approach rather than the less intuitive proposition-based approach, and I shall then attempt to provide some brief indication of the different ways in which the subject has been applied.

2. Language and Logic

2.1. Language

The language L of classical sentential logic may be taken to consist of the following symbols:
(i) the sentence letters p_1, p_2, p_3 ...;
(ii) the truth-functional connectives $\sim$, $\vee$, $\wedge$ and $\supset$;
(iii) the parentheses (and).

2.2. Formulas

The formulas of L are generated by means of the following rules:

(i) each sentence letter is a formula;
(ii) if A is a formula then so is $\sim A$;
(iii) if A and B are formulas, then so are $(A \vee B)$, $(A \wedge B)$ and $(A \supset B)$.

(see Section 5.1 in *Formal Logic*).

The language $L(\Box)$ of modal sentential logic is obtained by adding a modal operator $\Box$ for 'necessity' to the language L. There is one additional rule for generating formulas:

(iv) If A is a formula, then so is $\Box A$.

We adopt the following abbreviations:

$\Diamond A$ (‘possibly A’) for $\sim\Box\sim A$;

$A \rightarrow B$ (‘A strictly implies B’) for $\Box(A \supset B)$.

We follow standard notation in using ‘$\supset$’ for material implication and in using ‘$\rightarrow$’ for various stricter forms of implication. It should be noted that embeddings of $\Box$ are allowed. Thus $\Box\Box A$ or $\Box A \supset \Box\Box A$ or $\Box(\Box A \supset A) \supset \Box A$ are all formulas. A large part of modal logic is concerned with the way $\Box$ embeds.

2.3. Systems

A *modal system* is characterized by a set of axioms and rules. Say that a formula of $L(\Box)$ is *truth-functionally valid* if it is a substitution-instance of a tautology of L. The formula $(\Box A \vee \sim\Box A)$, for example, is truth-functionally valid since it is a substitution-instance of the tautology $(p_1 \vee \sim p_1)$. The axiom and rules for the *minimal* modal system K are then as follows:

Truth-functionality. All truth-functionally valid formulas;

Distribution. $\Box(A \supset B) \supset (\Box A \supset \Box B)$,

Modus Ponens. $A,\ A \supset B /\ B$

Necessitation. $A/\Box A$.

The *theorems of* the system K are all those formulas that can be obtained from the axioms by means of the rules. To be more exact, the set of K-theorems is the smallest set to contain the axioms of Truth-functionality and Distribution and to be closed under the rules of Modus Ponens and Necessitation (thus if A and $A \supset B$ are in the set then so is B and if is A is in the set then so is $\Box A$). We write ‘$\_{K}$ A’ to indicate that A is a theorem of K (and similar terminology will be adopted for other systems of modal logic). What motivates the choice of system is that it is the weakest to which semantic techniques described below will apply.

2.4. Meta-theorems

Let us note two basic results concerning the deductive power of K. They will be useful in establishing completeness below and are also often useful in establishing that particular formulas are theorems. Say that B is a *truth-functional consequence* of the set of formulas Δ if there are formulas $A_1, A_2, \ldots, A_n$ of Δ, $n \geq 0$, for which $(A_1 \supset (A_2 \supset \ldots \supset(A_n \supset B) \ldots))$ is truth-functionally valid; and say that B is a K-*deducible* from Δ if there are formulas $A_1, A_2, \ldots, A_n$ of Δ, $n \geq 0$, for which $(A_1 \supset (A_2 \supset \ldots \supset(A_n \supset B) \ldots))$ is a K-theorem. We write ‘$\Delta \vdash_K B$’ (or ‘$A_1, A_2, \ldots \vdash_K B$’ when $\Delta = \{A_1, A_2, \ldots\}$ or ‘$\Delta \vdash B$’ when the system is clear from the context) to indicate that A is K-deducible from Δ.

Lemma 1

(i) (TFC) Truth-functional consequences of K-theorems are K-theorems;

(ii) (Generalized Distribution) If $A_1, A_2, \ldots, A_n \vdash_K B$, then $\Box A_1, \Box A_2, \ldots, \Box A_n \vdash_K \Box B$.

Proof (i) Suppose that B is a truth-functional consequence of $A_1, A_2, \ldots, A_n$, i.e. $(A_1 \supset (A_2 \supset \ldots \supset (A_n \supset B) \ldots))$ is a K-theorem. If each of $A_1, A_2, \ldots, A_n$ is a K-theorem, it follows by n applications of modus ponens that B is a theorem.

(ii) Suppose that $A_1, A_2, A_3 \ldots, A_n \vdash_K B$, i.e. that $(A_1 \supset (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots)))$ is a K-theorem. By the rule of Necessitation $\Box (A_1 \supset (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots)))$ is a K-theorem; by the Distributivity axiom, $\Box (A_1 \supset (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots))) \supset (\Box A_1 \supset \Box (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots)))$ is a K-theorem; and so by modus ponens, $(\Box A_1 \supset \Box (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots)))$ is a K-theorem. We may show in a similar way that $\Box (A_2 \supset (A_3 \supset \ldots \supset (A_n \supset B) \ldots)) \supset (\Box A_2 \supset \Box (A_3 \supset \ldots \supset (A_n \supset B) \ldots))$ is a K-theorem; and so by part (i) of the lemma, $\Box A_1 \supset (\Box A_2 \supset \Box (A_3 \ldots \supset (A_n \supset B) \ldots))$ is a K-theorem. Continuing in this way, we may drive $\Box$ successively inwards through the implications, thereby establishing the K-theoremhood of $(\Box A_1 \supset (\Box A_2 \supset \ldots \supset (\Box A_n \supset \Box B) \ldots))$.

3. Semantics

We describe the possible worlds semantics for modal logic.

3.1. Models

A K-*model* M is an ordered triple (W, R, φ), where W is a non-empty set, R is a binary relation on W (i.e. $R \subseteq W \times W$), and φ is a function taking each sentence-letter $p_1, p_2, p_3 \ldots$ into a subset of W. Intuitively, W is the set of 'worlds', relative to which formulas are evaluated, R is an accessibility-relation on worlds, holding between worlds w and v when v is relevant to the evaluation of a necessity formula at w, and φ is a 'valuation' that specifies, for each sentence-letter p, the set of worlds $\varphi(p)$ in which p is true. An example of a K-model may be obtained by letting:

$W = \{0, 1, 2, \ldots\}$;
$R = \{(m, n): n = m + 1\}$; and
$\varphi = \{(p_n, \{n\}): n = 0, 1, 2, \ldots\}$.

Thus in this model, each sentence-letter p_n is true at exactly the one world n and the only world accessible from a given world m is its 'successor' n + 1. (We might think of the worlds as 'days', starting with day 0 and continuing into the infinite future; accessibility is the 'tomorrow' relation; and p_n is sentence stating that it is day n.)

3.2. Truth

Given a K-model $M = (W, R, \varphi)$, we may define when a formula is true at a world of the model. We use the notation '$w \vDash A$' to indicate that A is true at w (or '$w \vDash_M A$' if we wish to be explicit about the underlying model M). The definition is as follows:

M(i) $w \vDash p$ iff $w \in \varphi(p)$, for any sentence-letter p;

M(ii) $w \vDash {\sim}B$ iff it not the case that $w \vDash B$;

M(iii) $w \vDash B \vee C$ iff $w \vDash B$ or $w \vDash C$ (and similarly for $\wedge$ and $\in$);

M(iv) $w \vDash \Box B$ iff $v \vDash B$ whenever $w\,R\,v$.
It is readily determined on the basis of these clauses and the abbreviations that have been laid down that:

M(v) $w \vDash \Diamond B$ iff $v \vDash B$ for some v for which $w\,R\,v$; and

M(vi) $w \vDash B \to C$ iff $v \vDash C$ whenever $w\,R\,v$ and $v \vDash B$.

A possibility formula $\Diamond B$ is true at a world just in case the possibilized formula B is true at some accessible world; and a strict implication $B \to C$ is true at a world just in case the consequent C is true in any accessible world in which the antecedent B is true.

Let us illustrate with the example-model above (in which we might think of '$\Box$' as meaning *tomorrow*). The sentence-letter p_n will be true at the world n (i.e. $n \vDash p_n$) given that $n \in \varphi(p_n)$.

Since $n - 1\ R\ n$ for $n > 0$, it follows that $\Diamond p_n$ is true at n-1 (i.e. $n\text{-}1 \vDash \Diamond p_n$). Use $\Diamond^m$ for $\Diamond\Diamond \ldots \Diamond$ (m times). It then follows, for $n > m$, that $\Diamond^m p_n$ is true at $n - m$ (i.e. $n - m \vDash \Diamond^m p_n$). The reader may readily establish the converse, viz. that $k \vDash \Diamond^m p_n$ only if $k = n - m$; and so $\Diamond^m p_n \supset \Diamond^l p_n$ will be false at the world $n - m$ whenever $l \neq m$.

3.3. Validity

We say that a formula A is *true in* a model $M = (W, R, \varphi)$ if it true in every world $w \in W$ of the model; and we say that a formula A is K-*valid* if it is true in every model. We write $\vDash_K A$ to indicate that the formula A is K-valid. We may show, for example, that any instance $\Box(A \supset B) \supset (\Box A \supset \Box B)$ of the Distribution axiom is K-valid. For take any model $M = (W, R, \varphi)$ and world $w \in W$ and suppose that $w \vDash \Box(A \supset B)$ and $w \vDash \Box A$. We must then show that $w \vDash \Box B$. But if $w \vDash \Box(A \supset B)$ then $v \vDash B$ whenever $v \vDash A$ and $w\,R\,v$; and if $w \vDash \Box A$, then $v \vDash A$ whenever $w\,R\,v$. So $v \vDash B$ whenever $w\,R\,v$; and consequently $w \vDash \Box B$, as required. On the other hand, the formula $\Diamond^m p_n \supset \Diamond^l p_n$, for $n > m$ and $l \neq m$, is not K-valid since it is not true at the world $n - m$ in the example model above.

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Bibliography

[1] Anderson A. R., Belnap N. D. (1992) *Entailment: The Logic of Relevance and Necessity*, Princeton: Princeton University Press.

[2] Blackburn P., de Rijke M., Venema Y. (2002) *Modal Logic*, Cambridge: Cambridge University Press.

[3] Boolos G., (1993) *The Logic of Provability*, Cambridge: Cambridge University Press.

[4] Goedel K., {1933) *Eine Interpretation des Intuitionistischen Aussagenkalculus* in *Ergebnisse eines Mathematisichen Kolloquium* 3, 34-40.

[5] Harel D., Kotzen D., Tiuryn J., (2000) *Dynamic Logic*, Cambridge: MIT Press.

[6] Hilpinen R., (1970) *Deontic Logic: Introductory and Systematic Readings*, Dordrecht: D. Reidel

[7] Kripke S., (1959) *A Completeness Theorem in Modal Logic*, Journal of Symbolic Logic 24, 1-14.

[8] Lewis C. I., Langford C. H. (1918) *A Survey of Symbolic Logic*, California: University of California Press.

[9] Lewis D., (1973) *Counterfactuals*, Oxford: Blackwell.

[11] Prior A. N. (1967) *Past, Present and Future*, Oxford: Clarendon Press.

[12] Rosser J. B., Turquette A. R. (1952) *Many-valued Logics*, Amsterdam: North-Holland

[13] Tarski A., (1983) 'The Concept of Truth in Formalized Languages' in *Logic, Semantics and Metamathematics* (ed. P. Woodger), 2nd edition, Hackett Publishing 152-178.

[14] Tarski A. (1948) *Some Theorems about the Sentential Calculi of Lewis and Heyting*, Journal of Symbolic Logic 13, 1-15

Biographical Sketch

Kit Fine is Silver Chair in Philosophy and Mathematics at New York University. His research has principally been in the general areas of logic and philosophy. His most recent book, 'Limits of Abstraction' (OUP, 2002), attempts to determine to what extent abstraction principles of the sort proposed by Frege might be used as a basis for mathematics.