AN INTRODUCTION TO FINITE VOLUME METHODS

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Contents

Summary

We propose an elementary introduction to the finite volume method in the context of gas dynamics conservation laws. Our approach is founded on the advection equation, the exact integration of the associated Cauchy problem, and the so-called upwind scheme in one space dimension. It is then extended in three directions: hyperbolic linear systems and particularly the system of acoustics, gas dynamics with the help of the Roe matrix and two space dimensions by following the approach proposed by Van Leer. A special emphasis on boundary conditions is proposed all along the text.

1. Advection Equation and Method of Characteristics

1.1. Advection Equation.

We consider a given real number $a > 0$ and we wish to solve the so called advection equation of unknown function $u(x,t)$:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad t \geq 0, \quad x \in \mathbb{R}. \tag{0.0.1}$$

We first look to the homogenous coherence of the different terms of equation (1.1.1) with reference to their dimensions. On the one hand, the ratio $\frac{\partial u}{\partial t}$ is homogenous to the dimension $[u]$ of function $u(\bullet,\bullet)$ divided by the dimension $[t]$ of the time and we have: $\frac{\partial u}{\partial t} \sim \frac{[u]}{[t]}$. On the other hand the expression $a\frac{\partial u}{\partial x}$ is homogenous to the dimension $[a]$ of scalar *a* multiplied by the ratio $\frac{[u]}{[x]}$ and we have $a\frac{\partial u}{\partial x} \sim [a]\frac{[u]}{[x]}$. From equation (1.1.1), the two previous terms $\frac{\partial u}{\partial t}$ and $a\frac{\partial u}{\partial x}$ have the same dimension and we deduce from the previous formulae the equality: $\frac{1}{[t]} \sim \frac{[a]}{[x]}$. Then we have established that the constant *a* is homogenous to a **celerity**:

$$[a] \sim \frac{[x]}{[t]} \tag{0.0.1}$$

The Cauchy problem for the model equation (1.1.1) is composed by the equation (1.1.1) itself and the following initial condition:

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{0.0.2}$$

where $\mathbb{R} \ni x \mapsto u_0(x) \in \mathbb{R}$ is some given function. We observe that the solution of equation (1.1.1) is constant along the characteristic (straight) lines that satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a. \tag{0.0.3}$$

Proposition 1.1. The solution is constant along the characteristic lines.

Let $0 \leq \lambda \leq t$ be some given parameter and $u(\bullet,\bullet)$ a solution of equation (1.1.1). Then function $u(\bullet,\bullet)$ is constant along the characteristic lines, *i.e.*

$$u(x - a\lambda, t - \lambda) = u(x,t), \qquad \forall x,t,\lambda. \tag{0.0.4}$$

The proof of Proposition 1.1. is obtained as follows. We consider a fixed point (x,t) in space-time $\mathbb{R}\times[0,+\infty)$ and the auxiliary function $[0,t]\ni\lambda\mapsto v(\lambda)=u(x-a\lambda,t-\lambda)$. We have, due to the usual chain rule of differentiation:

$\dfrac{dv}{d\lambda}=\left[(-a)\dfrac{\partial u}{\partial x}-\dfrac{\partial u}{\partial t}\right](x-a\lambda,t-\lambda)=0$ if function $u(\bullet,\bullet)$ is the solution of the advection equation (1.1.1). Then $v(\lambda)$ does not depend on variable λ and we have in particular $v(\lambda)=v(0)$, which exactly expresses the relation (1.1.5). We have in particular for $\lambda=t:u(x,t)=u(x-at,0)=u_0(x-at)$ as illustrated on Figure 1 □

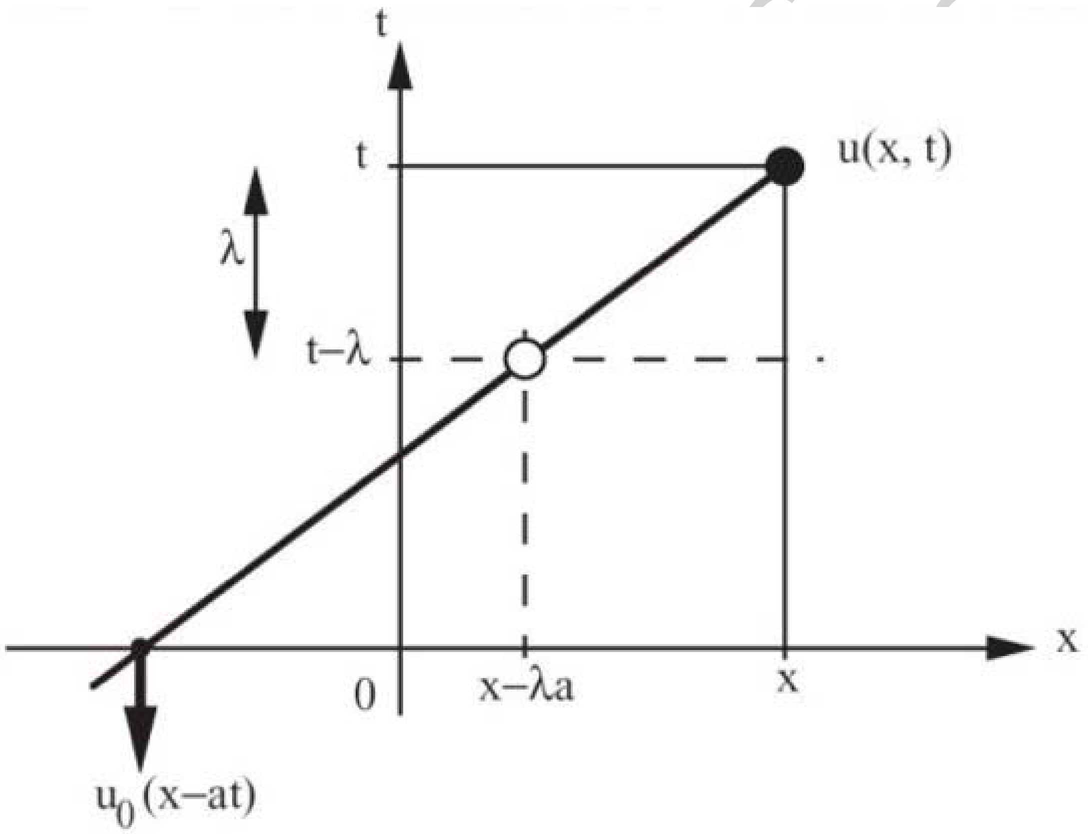

Figure 1: The solution $u(x,t)$ of the advection equation is constant along the characteristic lines.

1.2. Initial-Boundary Value Problems for the Advection Equation.

The second step is concerned with the so-called initial–boundary value problem considered for $x>0$ and $t>0$ with some given initial condition $u_0(x)$ for $t=0$ and a boundary condition $v_0(t)$ for $x=0$:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad t > 0, \quad x > 0, \quad \text{(equation)}$$
(1.2.1)

$$u(x,0) = u_0(x), \qquad x > 0, \qquad \text{(initial condition)} \tag{1.2.2}$$

$$u(0,t) = v_0(t), \qquad t > 0, \qquad \text{(boundary condition).} \tag{1.2.3}$$

Proposition 1.2. Advection in the quadrant: $x > 0$ **and** $t > 0$.

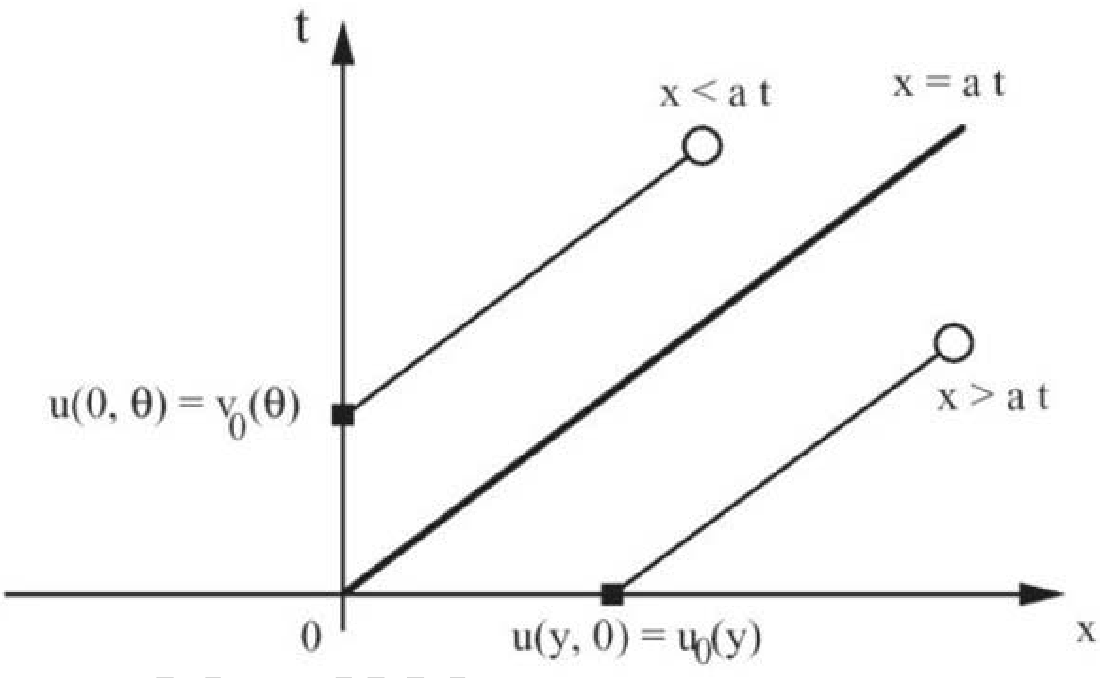

Figure 2: Initial- boundary value problem for the advection equation.

We suppose that $a > 0$. Then the solution of the advection equation (1.2.1) with the initial condition (1.2.2) and the boundary condition (1.2.3) is given by the relations

$$u(x,t) = u_0(x - at), \quad x - at > 0 \tag{1.2.4}$$

$$u(x,t) = v_0\left(t - \frac{x}{a}\right), \quad x - at < 0. \tag{1.2.5}$$

The initial condition $u_0(\bullet)$ is advected towards the space-time point (x,t) when $x - at > 0$ and the boundary condition $v_0(\bullet)$ is activated for $x - at < 0$.

Proof of Proposition 1.2.

In order to solve the problem (1.2.1)-(1.2.3), we use the method of characteristics. We fix a point (x,t) of space-time domain that satisfies $x>0, t>0$ and we go upstream in time with the help of the characteristic line that goes through this point (see Figure 2):

$$x(\lambda)=x-a\lambda, \quad t(\lambda)=t-\lambda. \qquad (1.2.6)$$

First case: $x-at>0$. When we take the particular value $\lambda=t$ in the previous relation (1.2.6), the particular point $y=x(t)=x-at$ on the axis of abscissa is strictly positive then the initial condition $u_0(y)$ is well defined. The solution $u(\bullet,\bullet)$ is constant on the characteristic line (see Proposition 1.1) that contains this particular point. Then relation (1.2.4) is established.

Second case: $x-at<0$. We consider the particular value $\lambda=\frac{x}{a}$ inside the expression (1.2.6). Then the corresponding foot of the characteristic belongs to the time axis: $\theta=t-\lambda=t-\frac{x}{a}$ and $\theta>0$ due to the inequalities $x<at$ and $a>0$. The solution is constant along the characteristic line going through this point and the relation (1.2.5) is established.

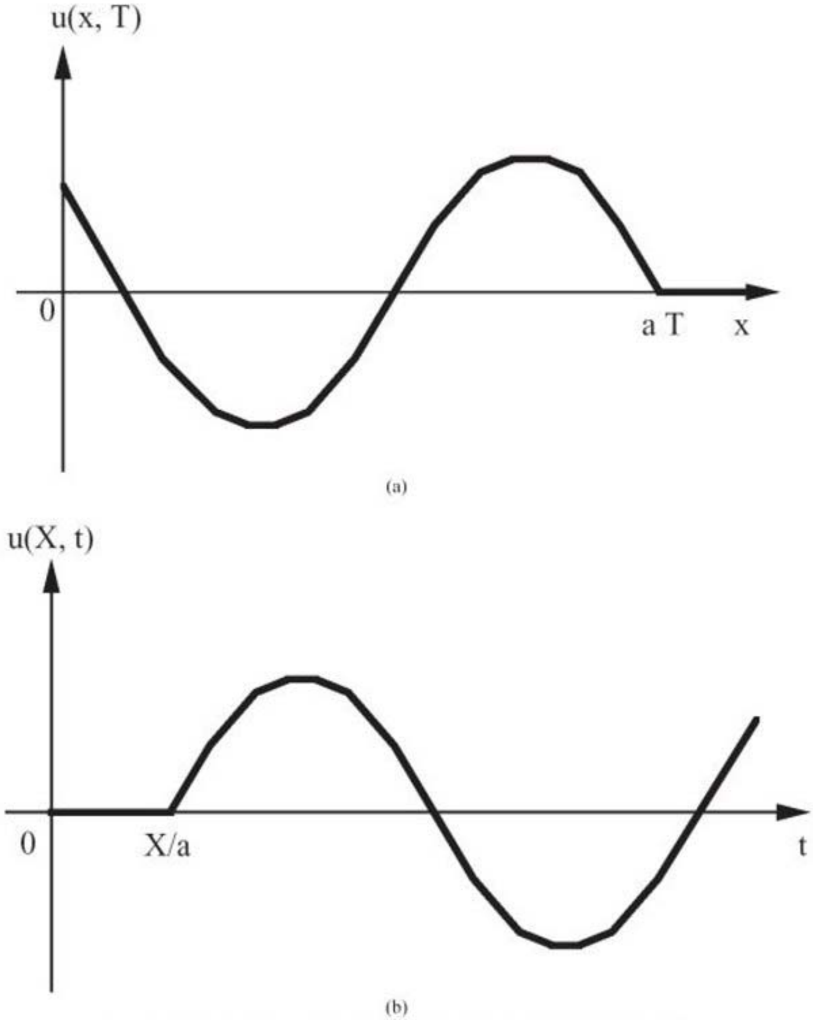

Figure 3: Snap shot of the solution of the advection equation at time $t=T$.

In the particular case where value $u_0(x)$ is identically equal to zero, i.e.,

$$u_0(x) = 0, \qquad x > 0, \tag{1.2.7}$$

and if the boundary condition $v_0(t)$ is sinusoidal for time positive to fix the ideas,

$$v_0(t) = \sin(\omega t), \quad t > 0, \tag{1.2.8}$$

the solution of the advection equation in the domain $x > 0, t > 0$ via the relations (1.2.4) and (1.2.5) can be considered with the following view points.

We take a snap shot of the solution $u(\bullet,\bullet)$ at a fixed time $T > 0.$ We consider the partial function $[0,+\infty) \ni x \mapsto u(x,T) \in \mathbb{R}$ and taking into account the relations (1.2.4), (1.2.5), (1.2.7) and (1.2.8), we have

$$u(x,T) = \begin{cases} \sin\left[\omega\left(T - \frac{x}{a}\right)\right] & x < aT \\ 0, & x > aT. \end{cases} \tag{1.2.9}$$

and this function is illustrated on Figure 3 (a).

We fix a particular position X in space and we look, as time is increasing, to the solution $u(\bullet,\bullet)$ at this particular point. We show on Figure3b the function $[0,+\infty] \ni t \mapsto u(X,t) \in \mathbb{R}$ and taking into account the relations (1.2.4), (1.2.5), (1,2,7) and (1.2.8), we have

$$u(X,t) = \begin{cases} 0, & t < \frac{X}{a} \\ \sin\left[\omega\left(T - \frac{x}{a}\right)\right] & t > \frac{X}{a}. \end{cases} \tag{1.2.10}$$

as shown in Figure 3(b)

1.3. Inflow and Outflow for the Advection Equation.

We still suppose that celerity *a* is positive and we consider the resolution of the advection (1.2.1) in the space-time domain

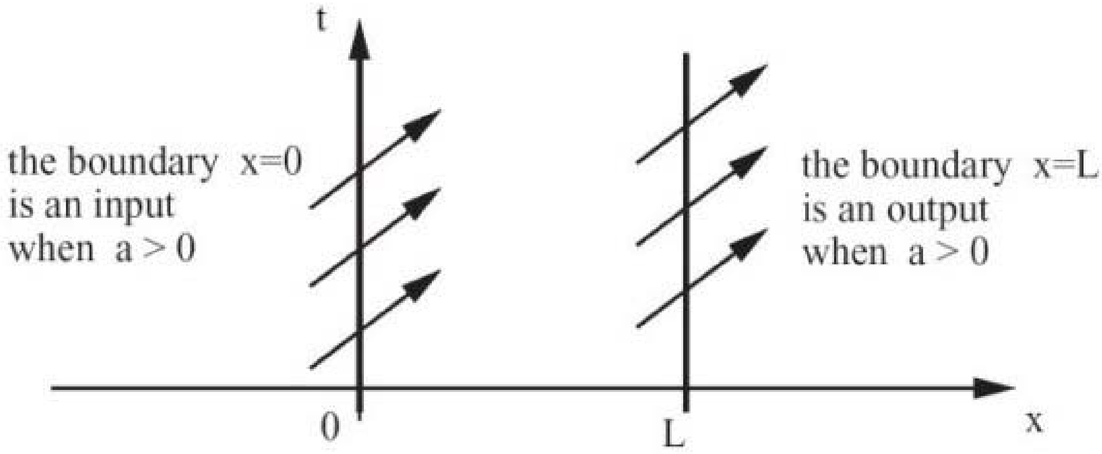

Figure 4: Initial-boundary value problem for the advection equation with $a>0$ in the domain $0<x<L$ and $t>0$.

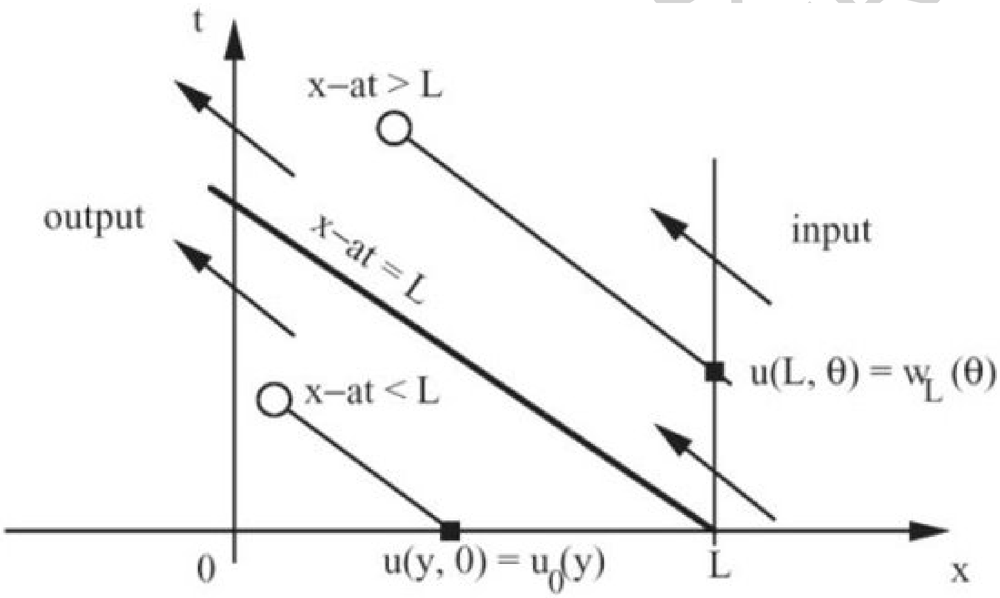

Figure5: Initial-boundary value problem for the advection equation with $a<0$ in the domain $0<x<L$ and $t>0$.

$$0<x<L, \qquad t>0. \tag{1.3.1}$$

The relations (1.2.4) and (1.2.5) can still be applied because the proof of Proposition 1.2 remains unchanged in this particular case. As a consequence of the previous property, we remark that **no boundary condition** is necessary at the particular position $x=L$ for solving the advection problem in the space-time domain defined in relations (1.3.1). The initial condition (1.2.2) has simply to be restricted in domain $[0,L]$:

$$u(x,0)=u_0(x), \qquad 0<x<L, \tag{1.3.2}$$

and the boundary condition (1.2.3) at $x=0$ remains unchanged :

$$u(0,t) = v_0(t), \qquad t > 0. \tag{1.3.3}$$

The difference between point $x = 0$ and point $x = L$ for the resolution of the advection equation in space-time domain (1.3.1) is due to the fact that we choose an orientation of the characteristic lines $x - at = \text{Constant}$ associated to an **increase** for the time direction. With this choice of time direction, the characteristic lines **enter** inside the space-time domain (1.3.1) at $x = 0$ and they go outside at $x = L$. The boundary condition (1.3.3) is given at the input of the domain (see Figure 4) and at $x = L$, there is a free output from space time domain (1.3.1), without necessity to specify any numerical boundary condition.

If we change the sign of celerity a, i.e., if we suppose now

$$a < 0, \tag{1.3.4}$$

the above analysis remains unchanged, but the algebraic relations (1.2.4) and (1.2.5) have to be modified (see Figure 5). We still start from relation (1.1.5) that expresses that the solution of the advection equation (1.1.1) is constant along the characteristics lines. The foot of the characteristic line that contains the particular point (x,t) in space- time is either the point $(y = x - at, 0)$ if $x - at < L,$ either the point $\left(L, \theta = t - \frac{1}{a}(x - L)\right)$ if $x - at > L.$ In the first case, we have $y > 0$ and $\theta < 0$ then the initial condition (1.3.2) is advected inside the domain (1.3.1) and we have :

$$u(x,t) = u_0(x - at), \qquad x - at < L. \tag{1.3.5}$$

On the contrary , if $x - at > L,$ we have $y > L$ and $\theta > 0$ then the boundary condition at $x = L$ that takes now the expression

$$u(L,t) = w_L(t), \qquad t > 0, \tag{1.3.6}$$

is advected inside the domain of study and we have:

$$u(x,t) = w_L\left(t + \frac{L}{a} - \frac{x}{a}\right), \qquad x - at > L. \tag{1.3.7}$$

We have established the following

Proposition 1.3. Advection in the domain $0 < x < L, a < 0.$

Under the hypothesis (1.3.4), the resolution of the advection equation (1.2.1) in the space-time domain (1.3.1) conducts to a **well posed** problem when we introduce the initial condition (1.3.2) on the interval $[0, L]$ and the boundary condition (1.3.6) at the **input** region located at $x = L$, without any boundary condition at the output located at $x = 0.$ The solution of Problem (1.2.1), (1.3.2) and (1.3.6) is given by the relations (1.3.5) and (1.3.7).

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Bibliography

[CDV92] D. Chargy, F. Dubois, J.P. Vila. Méthodes numériques pour le calcul d'écoulements compressibles, applications industrielles, *cours de l'Institut pour la Promotion des Sciences de l'Ingénieur*, Paris, septembre 1992.

[Ci78] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*, North Holland, Amsterdam, 1978.

[DM92] F. Dubois, O. Michaux. Solution of the Euler Equations Around a Double Ellipsoidal Shape Using Unstructured Meshes and Including Real Gas Effects, *Workshop on Hypersonic Flows for Reentry Problems*, (Désidéri- Glowinski-Périaux Editiors), Springer Verlag, vol. II, p. 358-373, 1992.

[DM96] F.Dubois, G.Mehlman. A non-parameterized entropy correction for Roe's approximate Riemann solver, *Numerische Mathematik*, vol 73, p.169-208, 1996.

[Du91] F. Dubois. Nonlinear Interpolation and Total variation Diminishing Schemes, *Third International Conference on Hyperbolic Problems*, (Engquist-Gustafsson Editors), Chartwell-Bratt, p. 351-359, 1991.

[Du2K] F. Dubois. Partial Riemann problem, Boundary conditions and gas dynamics, in *Artificial Boundary Conditions, with Applications to Computational Fluid Dynamics Problems*, (L. Halpern and L. Tourette Eds), Nova Science Publishers, 2000.

[FGH91] I. Faille, T. Gallouët, R.Herbin. Les Mathématiciens découvrent les Volumes Finis, *Matapli*, n° 23,pp. 37-48, octobre 1991.

[GR96] E. Godlewski, P.A.Raviart. *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Applied Mathematical Sciences, vol.118, Sprin-ger, New York, 1996.

[Go59] S.K. Godunov. A difference Method for the Numerical Computation of Discontinuous Solutions of the Equations of Fluid Dynamics, *Math. Sbornik*, vol.47, p.271-290, 1959.

[GZIKP79] S.K. Godunov, A.Zabrodine, M.Ivanov, A. Kraiko, G. Prokopov. *Résolution numérique des problémes multidimensionels de la dynamique des gaz*, Editions de Moscou, 1979.

[Ha83] A. Harten. High Resolution Schemes for Hyperbolic Conservation Laws, *Journal of Computational Physics*, vol.49, p.357-393, 1983.

[HLV83] A. Harten, P.D. Lax, B. Van Leer. On Upstream Differencing and Godunov-type Schemes for Hyperbolic Conservation Laws, *SIAM Review*, vol. 25, 1, p.35-61, January 1983.

[HGMW96] A.Hirschberg, J. Gilbert, R. Msallam, A.P.J. Wijnands. Shocks waves in trombones, *J. Acoust. Soc. Am.*, Vol.99, n°3, p. 1754-1758, 1996.

[Kr70] H.O.Kreiss. Initial Boundary Value Problems for Hyperbolic Systems, *Comm. Pure Applied Math.*, Vol.23, p.277-298, 1970.

[LL54] L. Landau, E. Lifchitz. *Fluid Mechanics*, Pergamon Press, 1954.

[MD99] R. Msallam, F. Dubois. Mathematical model for coupling a quasi-unidimensional perfect flow with an acoustic boundary layer, *Research report CNAM-IAT* n° 326/99, 1999.

[Pa80] S.V. Patankar. *Numerical Heat Transfer and Fluid Flow*, Hemisphere publishing, 1980.

[Po88] M. Pollet. Méthodes de calcul relatives aux interfaces missilespropulseurs, *Internal report*, Aerospatiale Les Mureaux, 1988.

[Roe81] P.Roe. Approximate Riemann Solvers, Parameter Vectors and Difference Schemes, *Journal of Computational Physics*, vol.43, p. 357-372,1981.

[Roe85] P. Roe. Some contributions to the Modelling of Discontinuous Flows, in *Lectures in Applied Mathematics*, vol.22, (Engquist Osher, Sommerville Eds), AMS, p. 163-193, 1985.

[RM67] R. D. Richtmyer, K.W. Morton. *Difference Methods for Initial Value Problems*, Interscience Publishing, J. Wiley & Sons, New York, 1967.

[VL77] B. Van Leer. Towards the Ultimate Conservative Difference Scheme IV. A New Approach to Numerical Convection, *Journal of Computational Physics*, vol.23, p. 276-299, 1977.

[VL79] B. Van Leer. Towards the Ultimate Conservative Differences Scheme V.A. Second Order Sequel to Godunov's Method, *Journal of Computational Physics*, vol 32. n°1, p. 101-136, 1979.

Biographical Sketch

François Dubois was born in Tours (France) in 1958. He obtained a Master degree in Theoretical Physics at the Ecole Nationale des Ponts et Chaussées. He received his PhD in Applied Mathematics at the University of Paris in 1988. He spent six years for the development of computational fluid dynamics in aerospace industry at EADS Les Mureaux. Since 1994, he is professor of Mathematics at the Conservatoire Nationale des Arts et Métiers (Paris), associate Professor at Ecole Polytenchnique (Palaiseau), and scientific advisor for EADS-France and the French "Commissariat de l'Energie Atomique". He is member of Société Mathématique de France, Société de Mathématiques Appliqueées et Industrielles and Association Française de Science des Systémes Cybernétiques Economiques et Techniques.