TU-GAMES

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Summary

This paper presents a perspective of TU-games. TU-games are cooperative games with transferable utility (TU), and each coalition's worth is given by a value. Thus TU-games are represented by a function that assigns each coalition a value it gains. The function is called a characteristic function. A main issue addressed in TU-games is how players divide—or should divide—the amount that they gain through cooperation. Thus competition among players arises since they want to gain as much as possible. This may involve negotiations, bargaining, threats, and so on, as well as different ideas of fairness. On the basis of these ideas various solution concepts have been proposed. Most solution concepts are defined in the imputation set—the set of payoff vectors satisfying total group rationality and individual rationality. In this paper, we present main solutions in TU-games: the core, stable sets, the bargaining set, the kernel, the nucleolus, and the Shapley value. For each solution we give its definition, explain its properties, and show how to find it by using simple examples. We then present typical applications of TU-games: applications to an exchange market, applications to decision making by voting, and applications to cost allocation and revenue allocation problems.

1. Introduction

There are two big streams in game theory: cooperative game theory and noncooperative game theory. In cooperative games players are allowed to cooperate; and if players agree to undertake joint action, their agreement is binding. Therefore, in cooperative game theory, coalition formation among players and distribution of the worth accrued from cooperation are the main interests. By contrast, in noncooperative games players behave independently and aim at their own goals. So the primary concern in noncooperative game theory is rationality in each player's decision making and outcomes produced through interactions of players' rational behavior.

Cooperative game theory is classified into two theories depending on the number of players. When there are two players, the questions are simply whether they cooperate and, given that they do, how to divide the worth they get through cooperation. If there are more than two players, we are faced with a more complex question—partial coalition formation, in which players may form a coalition against others. Thus even within cooperative games there exist two different theories: two-person cooperative game theory and n-person cooperative game theory. The former deals with bargaining between two players, and is therefore usually called a bargaining game, while the latter is called a coalitional form game. To describe how much each coalition gains, a function called a characteristic function is used, and so the latter is also called a characteristic function form game.

Since many coalitions can be formed among players, characteristic function form games are considerably complex. Hence a simplifying assumption of transferable utility is often used in these games. Suppose there exists a medium such as money that players can freely transfer among themselves. Suppose further that players' utility increases by one unit for every unit of the medium that they obtain. Such utility is called transferable utility. With transferable utility, each coalition can divide total utility among its members through transfer of the medium. Such a transfer is called a side payment. Thus a characteristic function assigns each coalition a number, in other words the total utility that it gains. These characteristic function form games are called transferable-utility games or TU-games. Without transferable utility, a characteristic function gives each coalition a set of utility vectors that it can get. Vector components are players' utility levels. These games are called nontransferable utility games or NTU-games. This paper deals with TU-games (for NTU-games see *Foundations of Noncooperative Games* and *NTU-Games*).

2. Characteristic Function Form Games

2.1. Characteristic Functions

A characteristic function form game is given by a pair (N, v) where $N = \{1,2, \ldots, n\}$ is a set of players and v is a real-valued function on the set of all subsets of N, called a characteristic function. A subset of N is called a coalition. The set N is called the grand coalition. For each coalition S, $v(S)$ represents the worth achievable by S, independent of players in the complement $N \setminus S$. For the empty set ϕ, we let $v(\phi) = 0$.

A characteristic function v is called superadditive if for all $S, T \subseteq N$ with $S \cap T = \phi$, $v(S) + v(T) \le v(S \cup T)$. The superadditivity means that two disjoint coalitions gain more by unifying into one coalition. Thus if a characteristic function is superadditive, then disjoint coalitions would merge to get more, and eventually a grand coalition would be formed. Furthermore, characteristic functions arising from real game situations usually satisfy superadditivity. Therefore, in characteristic function form games it has been usually assumed that a grand coalition N is formed. The primary concern is thus how to divide the worth $v(N)$ among players 1, …,n.

2.2. Imputations

Suppose players form a grand coalition and negotiate for how to share $v(N)$ among themselves. Let $(x_1, x_2, \cdots, x_n)$ be an n-dimensional vector of real numbers. The amount x_i denotes player i's payoff, and the vector is called a payoff vector.

Since $v(N)$ is shared among players, the equality $x_1 + x_2 + \cdots + x_n = v(N)$ must hold. The condition is called total group rationality or Pareto optimality. Furthermore, for each $i = 1, 2, \cdots, n$, $x_i \geq v(\{i\})$ must be satisfied. This condition is called individual rationality. If $x_i < v(\{i\})$ for some i, then player i can gain more by leaving the grand coalition and by playing alone. Thus individual rationality is necessary in order to keep every player in the grand coalition. A payoff vector $(x_1, x_2, \cdots, x_n)$ is called an imputation if it satisfies these two conditions. A set of imputations $\{x = (x_1, x_2, \cdots, x_n) \mid \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v(\{i\})\ \text{for each } i = 1,2,\cdots,n\}$ is denoted by A from here on. In characteristic function form games many solutions have been proposed. Most of them are defined in the imputation set.

In three-person games with $v(\{i\}) = 0$ for all i = 1,2,3, the imputation set is often depicted by an equilateral triangle with the height of $v(\{1,2,3\})$. The triangle is called a fundamental triangle (see Figure 1). Point x represents imputation (x_1, x_2, x_3) where x_1, x_2, x_3 are the length of perpendicular lines from x toward BC, CA, AB, respectively. Vertices A, B, C denote imputations (1,0,0), (0,1,0), (0,0,1), respectively.

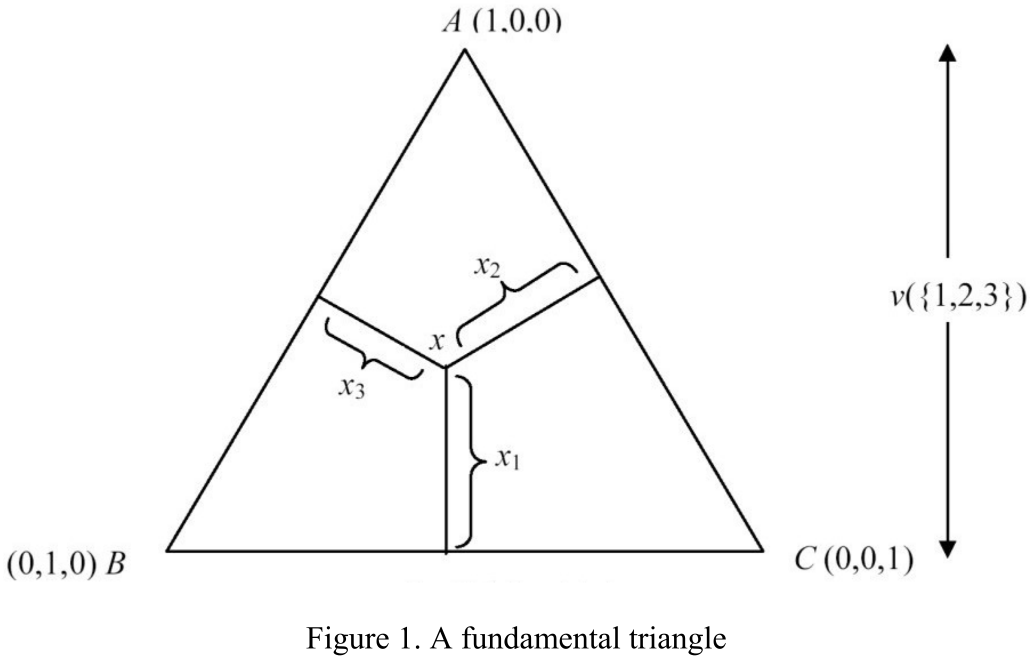

Figure 1. A fundamental triangle

2.3. Simple Examples

The three examples below will be used in the following sections to illustrate solutions:

- *Example 1*. Three players 1, 2, and 3 divide a joint revenue of 1 million yen by a simple majority vote. A coalition containing at least two players wins and secures the whole revenue.
- *Example 2*. Suppose, in Example 1, player 1 has a veto. Thus in order to win the vote a coalition must contain player 1.
- *Example 3*. Three neighboring towns 1, 2, and 3 plan to tap into a water resource for additional water supply. In the scenario that each town installs a water pipe independently, the estimated costs—in units of 10 million yen—are the following: 14 for town 1, 16 for town 2, and 20 for town 3. The estimated joint costs for two towns are 24 for towns 1 and 2, and 28 for towns 2 and 3. Towns 1 and 3 are unable to reduce construction costs—even if they cooperate—for geographical reasons. If all three towns cooperate the estimated joint cost is 30.

These examples are formulated as the following characteristic function form games:

Example 1. $N = \{1,2,3\}$

$$v(\{1,2,3\}) = 1,\ v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 1,\ v(\{1\}) \\ = v(\{2\}) = v(\{3\}) = v(\phi) = 0$$

Example 2. $N = \{1,2,3\}$

$$v(\{1,2,3\}) = 1,\ v(\{1,2\}) = v(\{1,3\}) = 1, v(\{2,3\}) \\ = 0,\ v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\phi) = 0$$

Example 3. $N = \{1,2,3\}$

$$v(\{1,2,3\}) = 20,\ v(\{1,2\}) = 6,\ v(\{1,3\}) \\ = 0,\ v(\{2,3\}) = 8,\ v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\phi) = 0$$

In Examples 1 and 2, coalitions that can gain 1 million yen are assigned the value 1, while other coalitions are assigned 0. In Example 3, the characteristic function above describes costs that coalitions can save. For example, if the three towns a install water pipe independently the total estimated cost is 14 + 16 + 20 = 50; while if they cooperate, the joint cost is 30. The three towns can save 50 – 30 = 20, and thus $v(\{1,2,3\}) = 20$. Values for other coalitions are determined in a similar manner.

3. Solutions

3.1. The Core

3.1.1. Coalitional Rationality and the Core

We will start with the most popular solution in characteristic function form games—the core. Recall the individual rationality defined in Section 2.2. It means that no player has an incentive to deviate from the grand coalition. The core requires in addition that no coalition deviates, that is, imputation $x=(x_1,x_2,\cdots,x_n)$ must satisfy the condition $\sum_{i\in S} x_i \ge v(S)$ for every nonempty coalition S $\subseteq$ N, S $\neq$ N. This condition is called coalitional rationality. The set of imputations satisfying coalitional rationality is called the core. Thus the core is given by the set $C=\{x=(x_1,x_2,\cdots,x_n)\in A \mid \sum_{i\in S} x_i \ge v(S) \text{ for every } S\subseteq N, S\neq N, S\neq \phi\}$. For each coalition S and each imputation x, let $e(S,x)=v(S)-\sum_{i\in S} x_i$. We call $e(S,x)$ the excess of S in x; then the core C is rewritten as $C=\{x=(x_1,x_2,\cdots,x_n)\in A \mid e(S,x)\le 0 \text{ for every } S\subseteq N, S\neq N, S\neq \phi\}$.

If an imputation in the core is proposed, every coalition gains at least the amount that it gets by itself. Thus every coalition is satisfied by the imputation. Since the implication of the core is easily understood, the core is used in many fields. For example, it is employed for finding a stable outcome in economic and social systems, and for finding a solution to cost (or benefit) allocation problems.

3.1.2. Cores in the Examples

Let us find the core in Example 1. Take an imputation $x=(x_1,x_2,x_3)$ in the core *C*. Then by the total group rationality and the individual rationality, we obtain:

$$x_1+x_2+x_3=v(\{1,2,3\})=1, \tag{1}$$
$$x_1\ge v(\{1\})=0,\ \ x_2\ge v(\{2\})=0,\ \ x_3\ge v(\{3\})=0$$

In addition, the coalitional rationality implies:

$$x_1+x_2\ge v(\{1,2\})=1,\ \ x_1+x_3\ge v(\{1,3\})=1,\ \ x_2+x_3\ge v(\{2,3\})=1 \tag{2}$$
$$x_1\ge v(\{1\})=0,\ \ x_2\ge v(\{2\})=0,\ \ x_3\ge v(\{3\})=0$$

Summing up three inequalities in (2), we obtain $x_1+x_2+x_3\ge 3/2$, which contradicts (1). Hence the core is an empty set.

In Example 2, an imputation $x=(x_1,x_2,x_3)$ in the core *C* must satisfy:

$$x_1+x_2+x_3=v(\{1,2,3\})=1,\ x_1\ge v(\{1\})=0,\ \ x_2\ge v(\{2\})=0,\ \ x_3\ge v(\{3\})=0$$
$$x_1+x_2\ge v(\{1,2\})=1,\ \ x_1+x_3\ge v(\{1,3\})=1,\ \ x_2+x_3\ge v(\{2,3\})=0$$
$$x_1\ge v(\{1\})=0,\ \ x_2\ge v(\{2\})=0,\ \ x_3\ge v(\{3\})=0$$

Hence the core consists of a single imputation (1, 0, 0).

In Example 3, we have the following inequalities for an imputation $x=(x_1,x_2,x_3)$ in the core:

$x_1 + x_2 + x_3 = v(\{1,2,3\}) = 20$, $x_1 \geq v(\{1\}) = 0$, $x_2 \geq v(\{2\}) = 0$, $x_3 \geq v(\{3\}) = 0$
$x_1 + x_2 \geq v(\{1,2\}) = 6$, $x_1 + x_3 \geq v(\{1,3\}) = 0$, $x_2 + x_3 \geq v(\{2,3\}) = 8$
$x_1 \geq v(\{1\}) = 0$, $x_2 \geq v(\{2\}) = 0$, $x_3 \geq v(\{3\}) = 0$

Figure 2 illustrates the core. The core is given by pentagon *DEFBG*; and thus in this example the core is a large set. Imputation $x = (x_1, x_2, x_3)$ on *FE* (respectively, on *DG*) satisfies $x_2+x_3 = v(\{2,3\}) = 8$ or $x_1 = 20 - 8 = 12$ (respectively, $x_1 + x_2 = v(\{1,2\}) = 6$ or $x_3 = 20 - 6 = 14$).

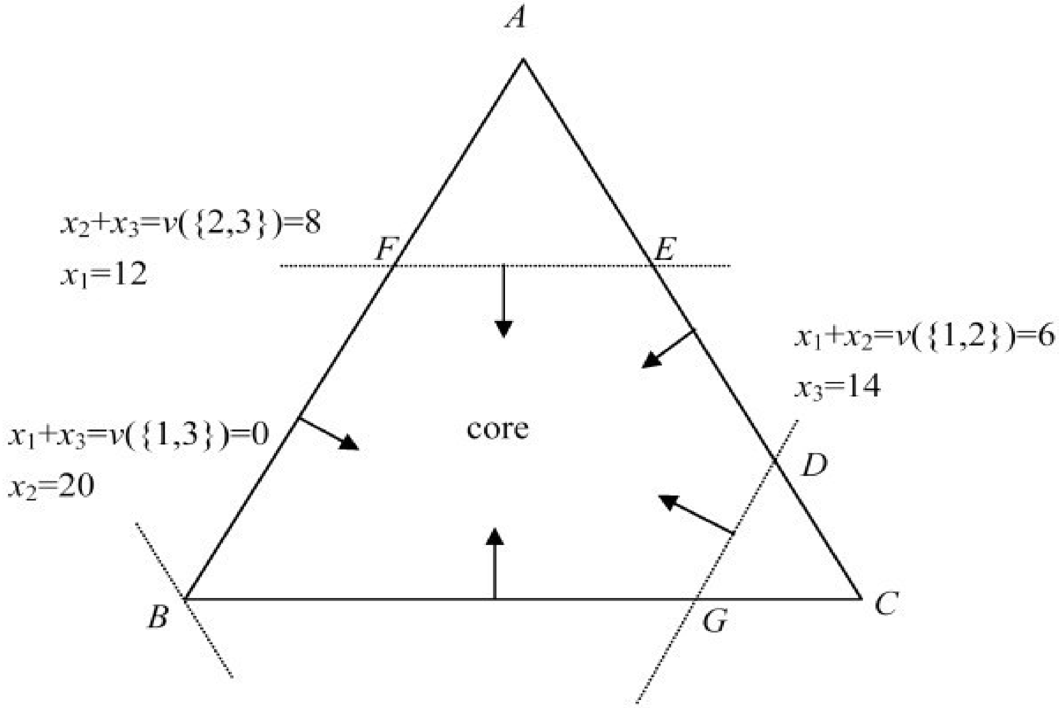

Figure 2. The core in Example 3

3.1.3. Conditions for Nonemptiness of the Core

The core is the set of payoff vectors $x = (x_1, x_2, \cdots, x_n)$ satisfying $\sum_{i\in N} x_i = v(N)$ and $\sum_{i\in S} x_i \geq v(S)$ for every $S \subseteq N$, $S \neq N$, $S \neq \phi$. Take a linear programming problem:

minimize $\sum_{i\in N} x_i$ subject to $\sum_{i\in S} x_i \geq v(S)$ for every $S \subseteq N$, $S \neq N$, $S \neq \phi$. (3)

Then the core is nonempty if and only if the optimal value of the problem is less than or equal to $v(N)$. Taking the dual of the problem, we obtain a necessary and sufficient condition that the core be nonempty. Take the dual of (3), that is:

$$\text{maximize} \sum_{\substack{S\subseteq N \\ S\neq N, S\neq\phi}} y_S v(S) \qquad (4)$$

subject to $\sum_{\substack{S \subseteq N \\ S \neq N, S \neq \phi, i \in S}} y_S = 1$ for all $i \in N$ and $y_S \geq 0$ for all $S \subseteq N, S \neq N, S \neq \phi$

Since both problems have feasible solutions, the duality theorem shows that the core is nonempty if and only if the optimal value of (4) is less than or equal to $v(N)$.

Take a family $\Gamma = \{S_1, \ldots, S_m\}$ of subsets of N. Then Γ is called balanced if there exist nonnegative numbers $y_1, \ldots, y_m$ such that $\sum_{\substack{j=1 \\ i \in S_j}}^{m} y_j = 1$ for all $i \in N$. The vector $y = (y_1, \ldots, y_m)$ is called a balanced vector. Then we may rewrite the above condition as follows: the core is nonempty if and only if for every balanced family $\Gamma = \{S_1, \ldots, S_m\}$, $\sum_{\substack{j=1 \\ i \in S_j}}^{m} y_j v(S_j) \leq v(N)$ .where $y = (y_1, \ldots, y_m)$ is a balanced vector of Γ. We may obtain a much simpler condition by taking a special balanced family. A balanced family $\Gamma = \{S_1, \ldots, S_m\}$ is called minimal if none of its subset is balanced. Then it is shown that the core is nonempty if and only if for every minimal balanced family $\Gamma = \{S_1, \ldots, S_m\}$ and its balanced vector $y = (y_1, \ldots, y_m)$, $\sum_{\substack{j=1 \\ i \in S_j}}^{m} y_j v(S_j) \leq v(N)$. For each minimal balanced family, its balanced vector is uniquely determined.

In three-person games with $N = \{1,2,3\}$, minimal balanced families are $\{\{1\},\{2\},\{3\}\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\}, \{1,3\}\}$, $\{\{3\}, \{1,2\}\}$, and $\{\{1,2\},\{1,3\},\{2,3\}\}$. The first three families are partitions, and thus components of their balanced vectors are all 1. Hence the condition for nonemptiness of the core clearly holds for these families if a game is superadditive. The balanced vector for the last family is (1/2, 1/2, 1/2). Therefore, the condition is $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \leq 2v(\{1,2,3\})$. It is easy to see that the condition is satisfied in Examples 2 and 3, but not in Example 1 (see *Linear Programming, Duality Theory*).

3.1.4. Dominance Relation and the Core

The core was originally defined through the notion of dominance relation. Take any two imputations $x, y \in A$ and any coalition S. If $\sum_{i \in S} x_i \leq v(S)$ and $x_i > y_i$ for all $i \in S$, then we say x dominates y via S, denoted $x \text{ dom}_S\, y$. The first condition indicates that the amount assigned to S in x can be gained by S itself; and the second condition shows that every member in S prefers x to y. Therefore, if imputation y is proposed, then coalition S can reject it by proposing imputation x. If there exists coalition S such that $x \text{ dom}_S\, y$, then we simply say x dominates y, denoted x dom y.

The set of imputations that are not dominated by any other imputation is called the core defined via dominance relation. Imputations in the core are stable, while imputations outside the core are unstable, since for each of them there is at least one coalition that can reject it. It is easily seen that the core defined via coalitional rationality is included in the core defined via dominance relation. The converse is not in general true; but if a game is superadditive the converse also holds.

3.2. Stable Sets

3.2.1. Definition

The stable set, the first solution in characteristic function form games, was defined by J. von Neumann and O. Morgenstern. It is defined through dominance relations. K, a set of imputations, is called a stable set if (1) for any two imputations x, y in K, neither x dom y nor y dom x, and (2) for any imputation z outside K, there exists an imputation x in K such that x dom z. The former is called internal stability, and the latter is called external stability.

As stated above, imputations outside the core (defined via dominance relation) are unstable in the sense that there is at least one imputation that dominates, that is, there exists at least one coalition that can induce a new imputation in which all members are better off. Now suppose the new imputation is also outside the core. Then the new imputation is unstable and thus the coalition may not reject the first imputation. Thus we would claim that a dominating imputation must be a stable one so that players may convince themselves to achieve the imputation when they reject an old one. The internal stability and the external stability of the stable set capture this point. Suppose the following are commonly understood among players. Imputations inside a stable set are "stable" in the sense that no coalition deviates from it; and imputations outside the set are "unstable" in the sense that at least one coalition deviates. Then the condition (1) implies that any imputation inside the set remains stable since no stable imputation dominates it; and the condition (2) implies that any imputation outside the set remains unstable since there exists at least one stable imputation that dominates. Thus we could say that the stable set gives global stability in the imputation set, while the core gives local stability.

3.2.2. Stable Sets in the Examples

Example 1 has two types of stable set. One consists of three points, (1/2,1/2,0), (1/2,0,1/2), and (0,1/2,1/2); and the other is a line segment $x_i = c,\ 0 \le c < 1/2,\ i = 1,2$ or 3. They are called a symmetric stable set and a discriminatory stable set, respectively. Figure 3 illustrates the former, while Figure 4 illustrates the latter where $i = 1$. As for the symmetric stable set, its internal stability is clear since dominations are done through two-person coalitions {1,2}, {1,3}, and {2,3}. To show the external stability, take an imputation $x = (x_1, x_2, x_3)$ different from the three points. Since $x_1 + x_2 + x_3 = 1$ and $x_1, x_2, x_3 \ge 0$, at least two elements must be less than 1/2. Without loss of generality assume $x_1, x_2 < 1/2$. Then (1/2, 1/2, 0) dominates (x_1, x_2, x_3) via coalition {1,2}. In Figure 3, imputations in *EFCD* (except line segments *ED*, *EF*) are dominated by (1/2, 1/2, 0) via coalition {1,2}.

We next show the stability of the discriminatory stable set. Take a line segment $x_1 = c$, 0

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Shigeo Muto is Professor of Mathematical Social Sciences in the Department of Value and Decision Science, Graduate School of Decision Science and Technology, Tokyo Institute of Technology. He received Master's and Doctoral degrees in Operations Research from Cornell University. He has done extensive research in the interface between game theory and social sciences to develop analyses in areas such as information economics, social dilemmas, and political decision making. Dr. Muto has more than 40 publications, including articles in scientific journals such as *International Journal of Game Theory*, *Games and Economic Behavior*, *Mathematics of Operations Research*, *Zeitschrift für Operations Research*, *Mathematical Social Sciences, Econometrica*, *European Economic Review*, *Journal of the Operations Research Society of Japan*, *and Economic Studies Quarterly.* He is on the editorial board of *International Journal of Game Theory*. Dr. Muto is a member of the Game Theory Society, the Operations Research Society of Japan, the Japanese Economic Association, and the Japanese Association for Mathematical Sociology.